Erdős–Straus Conjecture

Overview

This poster will cover
- Background
- Purpose
- Method
- Results/Benchmarks
- Further Research

of the Erdős–Straus conjecture

Background

The Erdős–Straus conjecture is an unsolved sum of unit fractions pertaining to number theory. It states that for all integers $n \geq 2$, there exist positive integers x, y, z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Although many expansions of sums of unit fractions exist, this one is of particular interest, as it continues to be an open problem, with verification only up until $n \leq 10^{14}$ as demonstrated in [1].

Purpose

The purpose of this study is to determine the validity of the Erdős–Straus conjecture, that is, whether solutions exist, up to $n = 10^{15}$. The hope is that these results will offer an easily checkable and reproducible verification of the E-S conjecture for the first time. In addition, it is our hope that these results can be used to detect patterns to allow further progress in closing this problem.

Method

Our program is written in C, utilizing the libraries MPI, GMP, and FLINT for parallelization, handling massive integers, and finding successive primes respectively. We test and run our implementation on Graham for performance reasons. Writing the algorithm, we added several optimizations that aren't obvious from a vanilla implementation of the solution. The first major improvement in speed comes from having a lower and upper bound for possible x and y values, eliminating the need for an exhaustive iteration of all possible values for each n. As it turns out these bounds are:

$$\frac{n}{4} + 1 \leq x \leq \frac{3n}{4}$$

and when $\frac{1}{y} + \frac{1}{z} = \frac{m}{n}$

$$\frac{n}{m} + 1 \leq y \leq \frac{2n}{m}$$

Further optimization is done by using the lemma that if the conjecture fails for some n, then n must fall under one of the 6 following congruence classes of primes shown in [2]:

$$\begin{array}{ll} n \equiv 1 \mod 840 & n \equiv 11^2 \mod 840 \\ n \equiv 13^3 \mod 840 & n \equiv 17^2 \mod 840 \\ n \equiv 19^2 \mod 840 & n \equiv 23^2 \mod 840 \end{array}$$

Finally, as a last reduction to required computational power, we implemented David Eppstein's Optimization. This technique, with some basic transformations, turns the problem of $\frac{1}{y} + \frac{1}{z} = \frac{m}{n}$ into $y = \frac{r+n}{m}$, where y is valid for all r that are divisors of n^2 such that $r < n$, and y is an integer. After finding y, it is easy to find the right z.

Results/Benchmarks

Due to the current state of the project, much information remains in a "TBD" state, however, some benchmarks and results are present for smaller n values. Solutions for $n = 10^8$ exceed 27390749735722746186336, demonstrating the need for GMP to perform arithmetic on integers this size. Memory requirements do not exceed 2GB per core when running the algorithm for $n = 10^8$ on 64 cores, to sustain the constant flood of results. Admittedly, some cores use very little, while others use the whole allocation of 2GB. Run times for the system with the same configuration hover around 30 minutes per core, with an expected increase of 10x for every 10x increase in n.

Further Research

Unexplored areas of the Erdős–Straus conjecture are obviously solutions for $n > 10^{15}$, and research into different patterns these solutions form, or if they exist at all. Of course, if a pattern is discovered it can be a great help to developing a conclusive proof for this conjecture.

References

[1] Graham, R. L. (2013), *Paul Erdős and Egyptian fractions.* Bolyai Society Mathematical Studies, 25, 289-309.

[2] A. Swett *http://math.uindy.edu/swett/esc.htm.* (accessed on 12/8/12).

[3] Mordell, Louis J. (1967), Diophantine Equations, Academic Press, pp. 287–290.

[4] Elsholtz, Christian (2001), "Sums of k unit fractions", Transactions of the American Mathematical Society.

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