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Niceness theorems

by

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Abstract.

MSCS:

Key words and key phrases:

1. Introduction and statement of the problems.

In this lecture I aim to raise a new kind of question. It appears that many important mathematical objects (including counterexamples) are unreasonably nice, beautiful and elegant. They tend to have (many) more (nice) properties and extra bits of structure than one would a priori expect.

The question is why this happens and whether this can be understood [1].

This lecture is about lots of examples of this phenomenon such as Daniel Kan's observation that a group carries a comonoid structure in the category of groups if and only if it is a free group, the Milnor-Moore and Leray theorems in the theory of Hopf algebras, Grassmann manifolds and classifying spaces, and especially the star example: the ring of commutative polynomials over the integers in countably infinite indeterminates. This last one occurs all over the place in mathematics and has more compatible structures that can be believed. For instance it occurs as the algebra of symmetric functions in infinitely many variables, as the cohomology and homology of the classifying space **BU**, as the sum of the representation rings of the symmetric groups, as the free lambda-ring on one variable, as the representing ring of the Witt vectors, as the ring of rational representation of GL_∞, ...

To start with here is a preliminary list of the kind of phenomena I have in mind.

- A. *Universal objects.* That is mathematical objects which satisfy a universal property. They tend to have:

a) a nice regular underlying structure

b) additional universal properties (sometimes seemingly completely unrelated to the defining universal property)

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[1] There is of course the "anthropomorphic principle" answer, much like the question of the existence of (intelligent) life in this universe. It goes something like this. If these objects wern't nice and regular we would not be able to understand and describe them; we can see/understand only the elegant and beautiful ones. I do not consider this answer good enough though there is something in it. So the search is also on for ugly brutes of mathematical objects.

Also this anthropomorphic argument raises the subsidiary question of why we can only understand/describe beautiful/regular things. There are aspects of (Kolmogorov) complexity and information theory involved here.

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- B. Objects with a great deal of compatible structure tend to have a nice regular underlying structure and/or additional nice properties: "Extra structure simplifies the underlying object".

- C. Nice objects tend to be large and inversely large objects of one kind or another tend to have additional nice properties. For instance, large projective modules are free (Hyman Bass).

- D. Extremal objects tend to be nice and regular. (The symmetry of a problem tends to survive in its extremal solutions is one of the aspects of this phenomenon; even (when properly looked at), when there is bifurcation (symmetry breaking) going on.)

- E. Uniqueness theorems and rigidity theorems often yield nice objects (and inversely). They tend to be unreasonably well behaved. I.e. if one asks for an object with such and such properties and the answer is unique the object involved tends to be very regular. This is not unrelated to D.

Concrete examples of all these kinds of phenomena will be given below (section 2) as well as a (pityful) few first explanatory general theorems (section 3).