Maximal cliques in the collinearity graphs of geometries of simplex codes

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Linear codes

Let $\mathbb{F}_q$ be the field consisting of $q = p^r$ elements, where p is a prime number. Consider the n-dimensional vector space $\mathbb{F}_q^n$ over this field.

Every k-dimensional subspace $C \subset \mathbb{F}_q^n$ is a *linear* $[n, k]_q$ *code* and its vectors are *codewords* of this code.

Any $(k \times n)$-matrix whose rows form a basis of C is called a *generator matrix* of C.

The code C is *non-degenerate* if every column of a generator matrix is non-zero.

Simplex codes

We assume that

- $n = [k]_q = \frac{q^k-1}{q-1}$, where $k \geqslant 2$, i.e. n is the number of points in $\mathrm{PG}(k-1, q)$,
- C is a non-degenerate linear $[n, k]_q$ code such that the columns in every generator matrix of C are mutually non-proportional.

In this case, we say that C is a *q-ary simplex code of dimension k*.

The Hamming weight of every non-zero codeword in such a code is equal to q^{k-1}, i.e. such codes are *equidistant*.

Any two q-ary simplex codes are equivalent, i.e. there is a monomial linear automorphism of $\mathbb{F}_q^n$ transferring one of them to the other.

Point-line geometry

A *point-line geometry* $(\mathcal{L}, \mathcal{P})$ is a pair, where $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines such that

- each line contains at least three points,
- the intersection of two distinct lines contains at most one point.

Distinct points are *collinear* if there is a line containing them.

The *collinearity graph* of $(\mathcal{L}, \mathcal{P})$ is a simple graph whose vertices are points and two points are connected by an edge if there is a line containing them.

A *subspace* is a subset $\mathcal{S} \subset \mathcal{P}$ such that for any collinear points $x, y \in \mathcal{S}$ the line joining x, y is contained in $\mathcal{S}$.

A subspace is *singular* if any two distinct points of this subspace are collinear.

Point-line geometry related to simplex codes

In $\mathrm{PG}(k-1,q)$ we consider the subgeometry $\mathcal{S}(k,q)$ whose points are 1-dimensional subcodes of q-ary simplex codes of dimension k and whose lines correspond to 2-dimensional subcodes of these simplex codes; such points and lines will be called *simplex*.

Singular subspaces of $\mathcal{S}(k,q)$ correspond to linear codes in $\mathbb{F}_q^n$ whose non-zero codewords are of Hamming weight q^{k-1}. q-ary simplex codes of dimension k are linear codes in $\mathbb{F}_q^n$ maximal with respect to this property.

Maximal singular subspaces correspond to q-ary simplex codes of dimension k.

Example

Let $k = 2$. Then $n = q + 1$ and

$$\langle(x_1, \ldots, x_{q+1})\rangle$$

is a simplex point if and only if precisely one of its coordinates is 0.

Consider simplex points $X = \langle(0, 1, 1, \ldots, 1)\rangle$ and $Y = \langle(1, 0, x_3 \ldots, x_{q+1})\rangle$. X and Y are collinear if and only if in the matrix

$$\begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & x_3 & \ldots & x_{q+1} \end{bmatrix}$$

all elements $x_3, \ldots, x_{q+1}$ are distinct.

Example

Now, consider simplex points

$$V = \langle (v_1, \dots, v_{q+1}) \rangle \text{ and } W = \langle (w_1, \dots, w_{q+1}) \rangle.$$

V and W are collinear if in the matrix

$$\begin{bmatrix} v_1 & \dots & v_{q+1} \\ w_1 & \dots & w_{q+1} \end{bmatrix}$$

all columns are mutually non-proportional (each of these columns is proportional to precisely one column from the previous martix).

If two simplex points have the same coordinate equal to 0, then they are not collinear.

Lemma

Two simplex points $\langle v \rangle$ and $\langle w \rangle$ are collinear in $\mathcal{S}(k, q)$ if and only if the $(2 \times n)$-matrix whose rows are v and w satisfies the following conditions: it contains precisely $[k-2]_q$ zero columns and every non-zero column is proportional to precisely q^{k-2} columns including itself.

Examples of geometries

Since there is the unique binary simplex code of dimension 2, $\mathcal{S}(2,2)$ is a line consisting of three points.
In the general case, the geometry $\mathcal{S}(k,2)$ can be identified with the geometry whose points are 2^{k-1}-element subsets of an n-element set and whose lines are triples $\{X, Y, X \triangle Y\}$, where X, Y are 2^{k-1}-element subsets whose intersection contains precisely 2^{k-2} elements and $X \triangle Y$ is their symmetric difference

The geometry $\mathcal{S}(2,3)$ is a (4×4)-grid.

The geometry of $\mathcal{S}(2,4)$ is more complicated. It was completely described in

M. Kwiatkowski, M. Pankov, *The graph of 4-ary simplex codes of dimension* 2, Finite Fields Appl. 67 (2020), 101709.

Cliques of collinearity graphs

The *collinearity graph* of $(\mathcal{L}, \mathcal{P})$ is a simple graph whose vertices are points and two points are connected by an edge if there is a line containing them.

Fisher's inequality gives the following.

Proposition

Every clique of the collinearity graph of $\mathcal{S}(k, q)$ contains no more than n elements.

Remark

If $q = 2$, then $n = 2^k - 1$ and there is a one-to-one correspondence between $(2^k - 1)$-cliques of the collinearity graph and symmetric $(2^k - 1, 2^{k-1}, 2^{k-2})$-designs.

Cliques of collinearity graphs

Every n-clique (consisting of n elements) in the collinearity graph of $\mathcal{S}(k,q)$ is maximal.

Every maximal singular subspace of $\mathcal{S}(k,q)$ consists of $n=[k]_q$ elements, so it is an n-clique.

In the paper

M. Kwiatkowski, M. Pankov, *On maximal cliques in the graph of simplex codes*, J. Geom. 115 (2024) 10

was proved that if one of the conditions

$$k=2 \text{ and } q=2,3,4 \text{ or } k=3 \text{ and } q=2$$

is satisfied, then every maximal clique of the collinearity graph is a maximal singular subspace.

We construct a class of n-cliques of the collinearity graph of $\mathcal{S}(k,q)$, $q\geqslant 5$ distinct from maximal singular subspaces.

F_s transformation

Let $q \geqslant 5$. Consider the transformation F_s of $\mathbb{F}_q^n$ sending every $x = (x_1, \ldots, x_n)$ to $x^s = (x_1^s, \ldots, x_n^s)$.

F_s is bijective if and only if the s-th power transformation of $\mathbb{F}_q$ (i.e. the transformation $a \to a^s$) is bijective, equivalently, if $\mathrm{GCD}(s, q-1) = 1$.

If $s = p^m$, then the s-th power transformation $a \to a^s$ is an automorphism of $\mathbb{F}_q$ and F_s is a monomial semilinear automorphism of $\mathbb{F}_q^n$.

Theorem

Theorem

If F_s is bijective $(i.e.\ \mathrm{GCD}(s, q-1) = 1)$ and $s \neq p^m$, then it sends every maximal singular subspace of $\mathcal{S}(k, q)$ to an n-clique of the collinearity graph of $\mathcal{S}(k, q)$, where any three mutually distinct points are not on a line of $\mathrm{PG}(n-1, q)$ and, consequently, this clique is not a singular subspace of $\mathcal{S}(k, q)$.

Normal rational curves

Suppose that $k = 2$. Then $n = [2]_q = q + 1$. Maximal singular subspaces of the geometry $\mathcal{S}(2, q)$ are lines corresponding to q-ary simplex codes of dimension 2 (lines contains precisely $q + 1$ points).

Recall that a *normal rational curve* in $\mathrm{PG}(m - 1, q)$ is the image of

$$\{\langle 1, t, \ldots, t^{m-1} \rangle : t \in \mathbb{F}_q\} \cup \{\langle (0, \ldots, 0, 1) \rangle\}$$

under a semilinear automorphism of $\mathbb{F}_q^m$. This is an *arc*, i.e. any mutually distinct $m + 1$ points from this subset form a basis of $\mathrm{PG}(m - 1, q)$.

Normal rational curves

Theorem

Suppose that F_s *is bijective* $(\mathrm{GCD}(s, q-1) = 1)$ *and the following condition is satisfied:*

(A) *the* p*-cyclotomic coset containing* s *contains also* $up^m - 1$ *with* $0 < u < p$.

Then F_s *sends every line of* $\mathcal{S}(2, q)$ *to a normal rational curve in an* s*-dimensional projective space over* $\mathbb{F}_q$.

Recall that s, s' belong to the same p-cyclotomic coset when $u' = up^m$ for $\{u, u'\} = \{s, s'\}$.

Thank you for your attention!