2-designs admitting a flag-transitive automorphism group

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Definition

A 2-(v, k, λ) **design** $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ consists of a set $\mathcal{P}$ of v **points**, and a set $\mathcal{B}$ of k-element subsets of $\mathcal{P}$, called **blocks**, such that every pair of distinct points is contained in exactly λ blocks.

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- $\mathcal{D}$ is **non-trivial** if $2 < k < v - 1$.
- A **flag** is any incident point-block pair of $\mathcal{D}$.

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An **automorphism** of $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation of the point-set $\mathcal{P}$ preserving the block-set $\mathcal{B}$. The set of all automorphisms of $\mathcal{D}$ is a group, called **the full automorphism group of $\mathcal{D}$**, denoted by $Aut(\mathcal{D})$.

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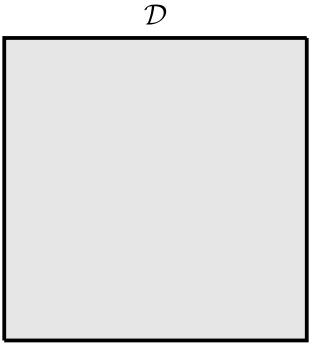
We are interested in the case where G acts flag-transitively on $\mathcal{D}$.

Flag-transitivity⇒Block-transitivity⇒Point-transitivity

Let $G \leq Aut(\mathcal{D})$, then



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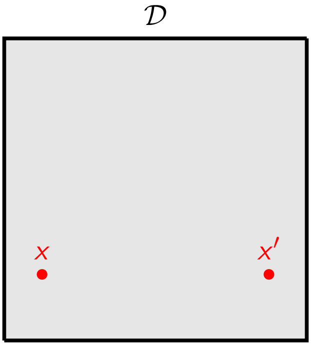

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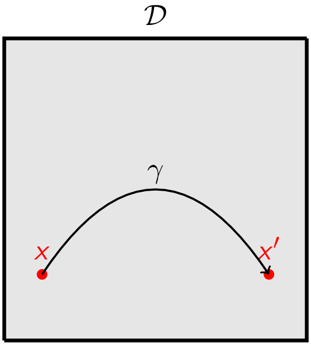

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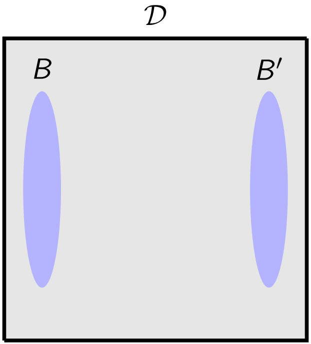

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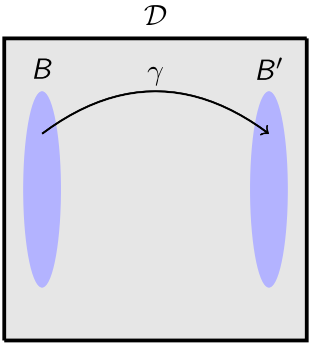

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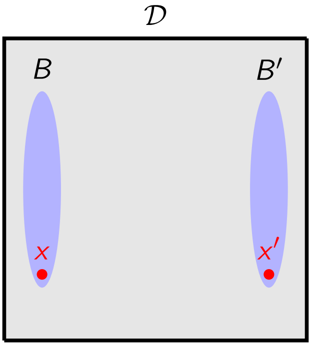

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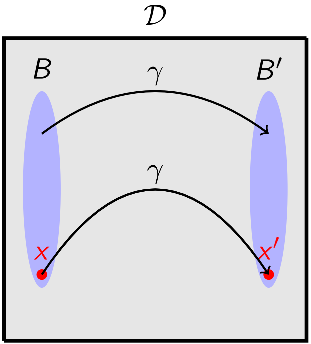

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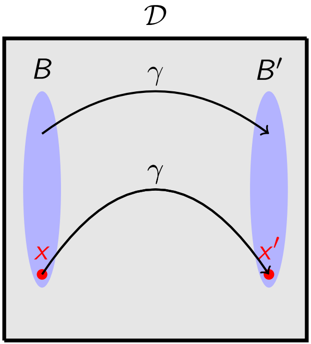

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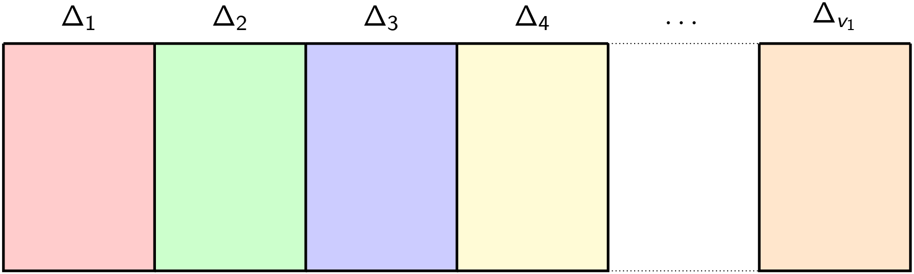

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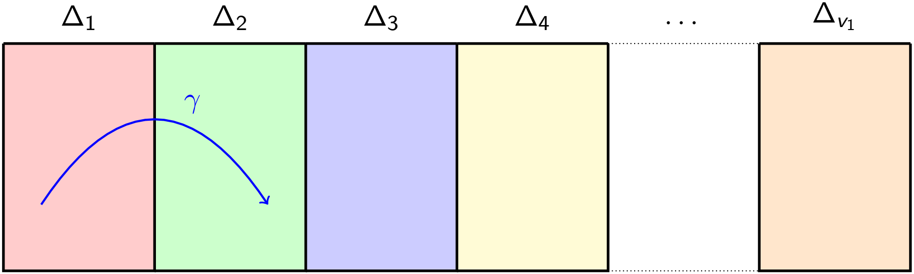

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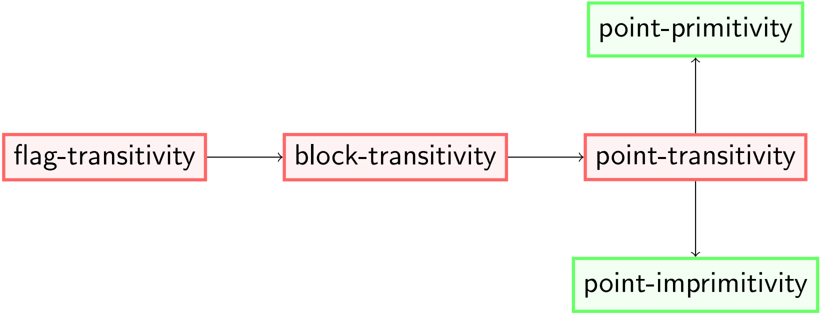

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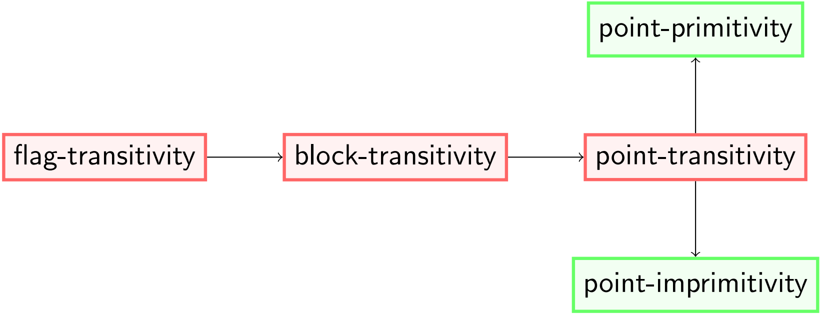

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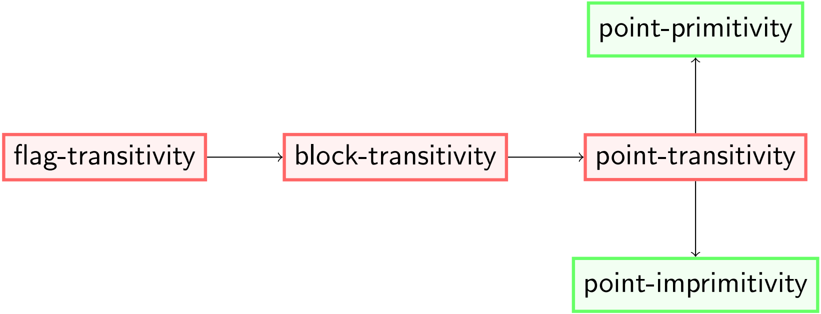

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The Higman-McLaughlin Theorem

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Theorem (Higman-McLaughlin, 1961)

Any flag-transitive automorphism group of a 2-design with $\lambda = 1$ acts point-primitively.

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- $\mathcal{D}$ is one of the two $(16, 6, 2)$ biplanes with G isomorphic to $(Z_2)^4 : S_4$ or $(Z_2 \times Z_8).(S_4.Z_2)$, respectively, (Husain (1945) and, independently, by Nandi (1946), and O'Relly-Reguerio (2005));

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Theorem (Davies, 1987)

For any fixed λ, there are only finitely many 2-(v, k, λ) designs with a flag-transitive point-imprimitive automorphism group.

Conditions ensuring point-primitivity

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Theorem

Let G be any flag-transitive automorphism group of a 2-(v, k, λ) design $\mathcal{D}$. Then G acts point-primitively on $\mathcal{D}$, provided that at least one of the following conditions on the parameters of $\mathcal{D}$ holds:

Line	Condition	Author(s)
1	$\lambda > (r, \lambda) \cdot ((r, \lambda) - 1)$	Dembowski, 1968, or
2	$(r, \lambda) = 1$	Kantor, 1969
3	$(r - \lambda, k) = 1$	
4	$r > \lambda(k - 3)$	
5	$(v - 1, k - 1) = 1$ or 2	
6	$k > 2\lambda^2(\lambda - 1)$	Devillers-Praeger, 2021
7	$v > \left(2\lambda^2(\lambda - 1) - 1\right)^2$	
8	$(v - 1, k - 1)^2 \leq v - 1$	Zhong-Zhou, 2023
9	$(v - 1, k - 1) = 3$ or 4	
10	k prime	

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Theorem (Devillers-Praeger, 2024)

Let G be any flag-transitive point-imprimitive automorphism group of a 2-(v, k, λ) design $\mathcal{D}$. If $v < 100$ and $\lambda \leq 4$, then one of the following holds:

1. $\mathcal{D}$ is one of the two $(16, 6, 2)$ biplanes with G isomorphic to $(Z_2)^4 : S_4$ or $(Z_2 \times Z_8).(S_4.Z_2)$;
2. $\mathcal{D}$ is the 2-$(45, 12, 3)$ design and $G \leq A\Gamma L_1(3^4)$;
3. $\mathcal{D}$ is the 2-$(15, 8, 4)$ design and $A_5 \trianglelefteq G \leq S_5$;
4. $\mathcal{D}$ is one of the two 2-$(16, 6, 4)$ designs;
5. $\mathcal{D}$ is the 2-$(36, 6, 4)$ design;
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There are no known examples corresponding to case (3).

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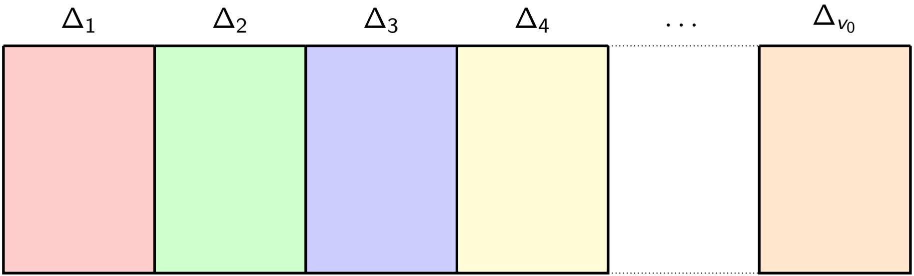

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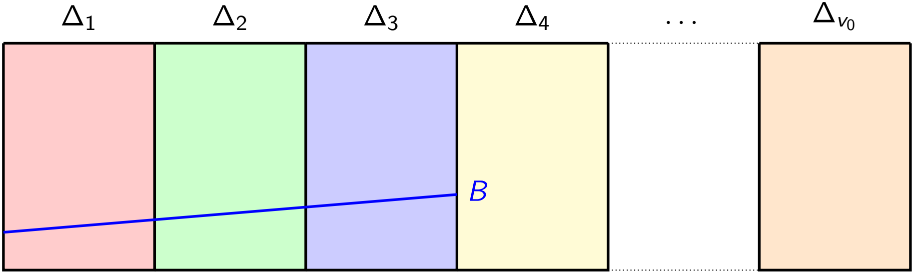

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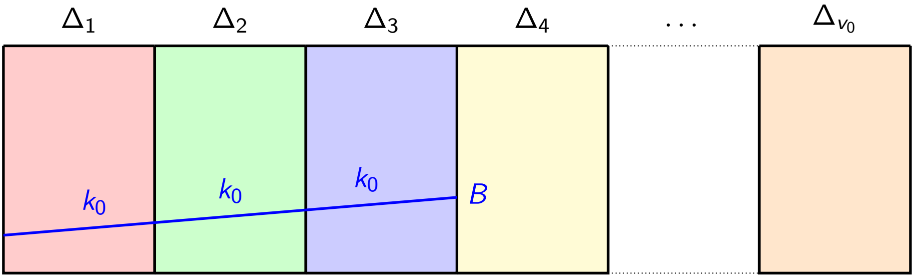

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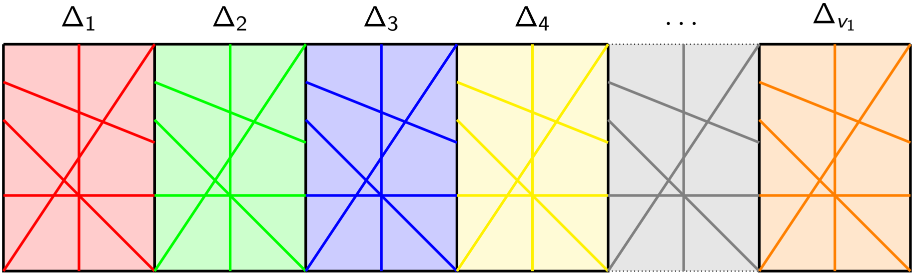

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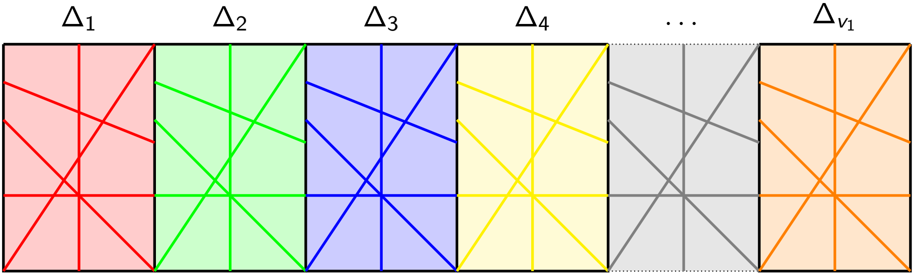

- For each $i = 1, ..., v_1$ the incidence structure $\mathcal{D}_{\Delta_i} = (\Delta_i, \mathcal{B}_{\Delta_i})$, where $\mathcal{B}_{\Delta_i} = \{B \cap \Delta_i \neq \varnothing : B \in \mathcal{B}\}$, is a 2-$(v_0, k_0, \lambda_0)$ design;

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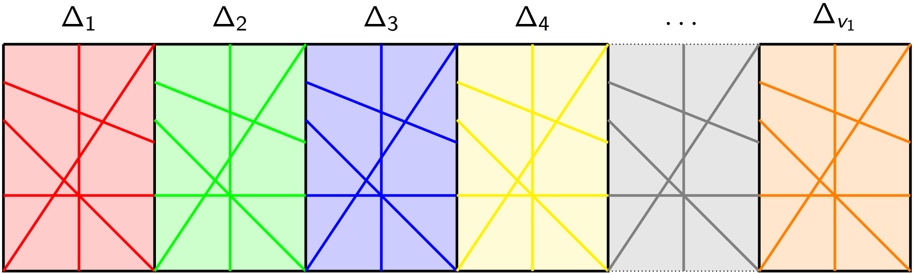

- For each $i = 1, ..., v_1$ the incidence structure $\mathcal{D}_{\Delta_i} = (\Delta_i, \mathcal{B}_{\Delta_i})$, where $\mathcal{B}_{\Delta_i} = \{B \cap \Delta_i \neq \varnothing : B \in \mathcal{B}\}$, is a 2-$(v_0, k_0, \lambda_0)$ design;
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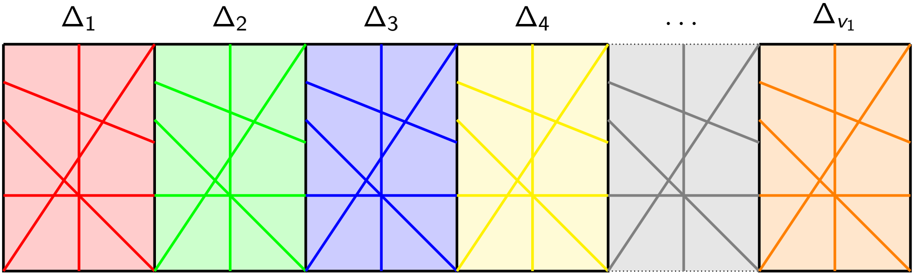

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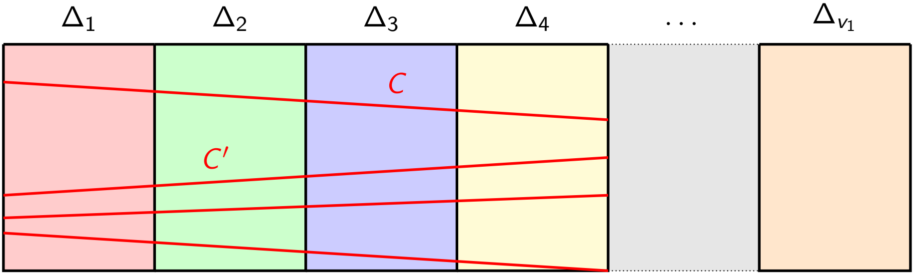
As the incidence structures corresponding to distinct classes $\Delta_i, \Delta_j \in \Sigma$ are isomorphic under elements of G mapping Δ_i to Δ_j we refer to $\mathcal{D}_{\Delta_i}$ as $\mathcal{D}_0$.

A fundamental Tool: the Theorem of Camina-Zieschang



1. for each block C of $\mathcal{D}$ the set $C(\Sigma) = \{\Delta \in \Sigma : C \cap \Delta \neq \varnothing\}$ has a constant size $k_1 = \frac{k}{k_0}$;
2. Let $\mathcal{B}^\Sigma$ be the quotient set defined by the equivalence relation $\mathcal{R} = \{(C, C') \in \mathcal{B} \times \mathcal{B} : C(\Sigma) = C'(\Sigma)\}$ on $\mathcal{B}$;
3. the incidence structure $\mathcal{D}_1 = \left(\Sigma, \mathcal{B}^\Sigma, \mathcal{I}\right)$, where $\mathcal{I} = \{(\Delta, C^\Sigma) \in \Sigma \times \mathcal{B}^\Sigma : \Delta \in C(\Sigma)\}$, is a 2-$(v_1, k_1, \lambda_1)$ design;
4. G^Σ acts flag-transitively on $\mathcal{D}_1$.

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- match the results obtained on $(\mathcal{D}_0, G_{\Delta}^{\Delta})$ and on $(\mathcal{D}_1, G^{\Sigma})$.

The Symmetric Case

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Theorem (Praeger-Zhou, 2006)

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be any symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive automorphism group G preserving a nontrivial partition Σ of $\mathcal{P}$ with v_1 classes of size v_0, where $v = v_0 v_1$. If $k > \lambda(\lambda - 3)/2$, then admissible parameters for $\mathcal{D}, \mathcal{D}_0$ and $\mathcal{D}_1$ are as in the following table:

Table: Admissible parameters for $\mathcal{D}, \mathcal{D}_0$ and $\mathcal{D}_1$

Line	v	k	v_0	k_0	v_1	k_1
1	$\lambda^2(\lambda+2)$	$\lambda(\lambda+1)$	λ^2	λ	$\lambda+2$	$\lambda+1$
2			$\lambda+2$	2	λ^2	$\lambda(\lambda+1)/2$
3	$\frac{\lambda+2}{2}\frac{\lambda^2-2\lambda+2}{2}$	$\frac{\lambda^2}{2}$	$\frac{\lambda+2}{2}$	2	$\frac{\lambda^2-2\lambda+2}{2}$	$\frac{\lambda^2}{4}$
4	$(\lambda+6)\frac{\lambda^2-2+4\lambda-1}{4}$	$\frac{\lambda(\lambda+5)}{2}$	$\lambda+6$	3	$\frac{\lambda^2-2+4\lambda-1}{4}$	$\frac{\lambda(\lambda+5)}{6}$

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1. $\mathcal{D}$ is one of the two $(16, 6, 2)$ biplanes with G isomorphic to $(Z_2)^4 : S_4$ or $(Z_2 \times Z_8).(S_4.Z_2)$;
2. $\mathcal{D}$ is the 2-$(45, 12, 3)$ design and $G \leq A\Gamma L_1(3^4)$;
3. $\mathcal{D}$ is the 2-$(15, 8, 4)$ design and $A_5 \trianglelefteq G \leq S_5$;
4. $\mathcal{D}$ is one of the four 2-$(96, 20, 4)$ designs (several possibilities for G).

Affine Resolvable Designs

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Definition

A **resolution** of a 2-design $\mathcal{D}_0 = (\mathcal{P}_0, \mathcal{B}_0)$ is any partition of $\mathcal{B}_0$ into sets, called **parallel classes**, each of which is a partition of $\mathcal{P}_0$.

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Definition

A resolvable 2-design $\mathcal{D}_0$ in which blocks in different classes have the same number of points in common is called **affine resolvable**.

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- Latin squares and Quasigroups are equivalent objects.

Camina-Zieschang Thm. & Cameron-Praeger Construction

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Symmetric 2-$(v_0 v_1, k_0 k_1, \lambda)$ design $\mathcal{D}$
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2-(v_0, k_0, λ_0) design $\mathcal{D}_0$
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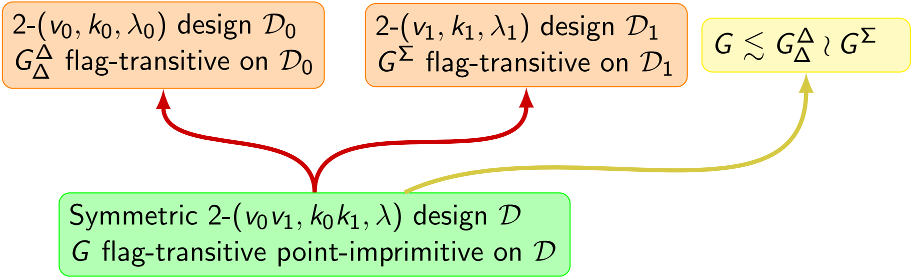

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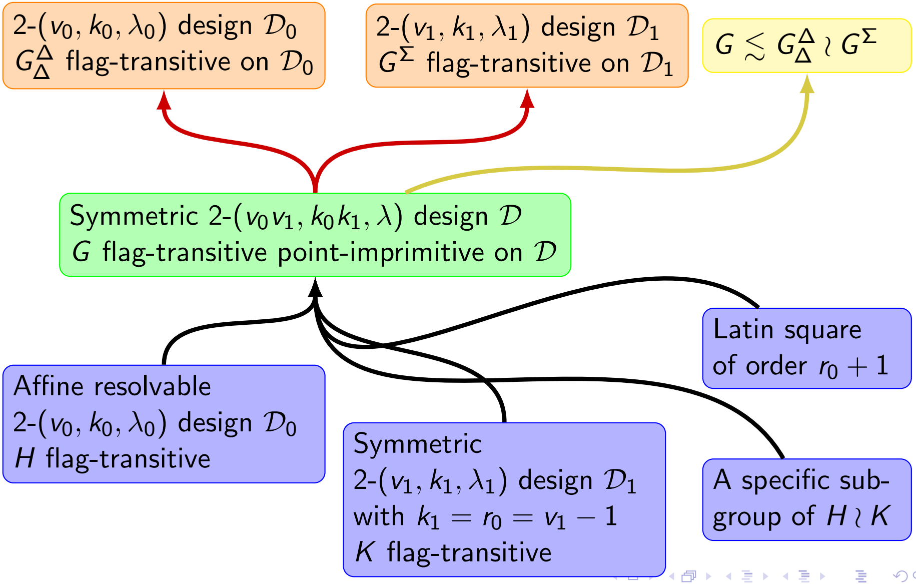

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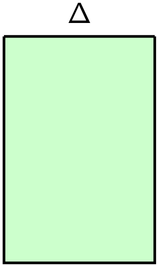

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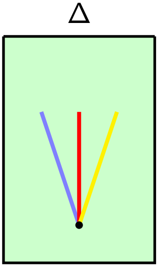

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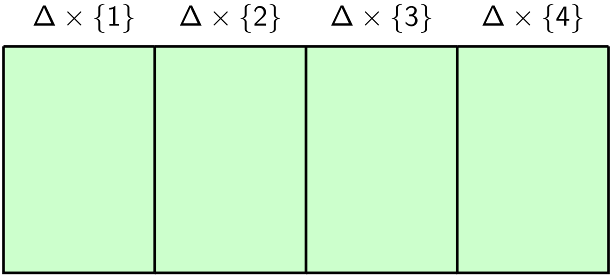

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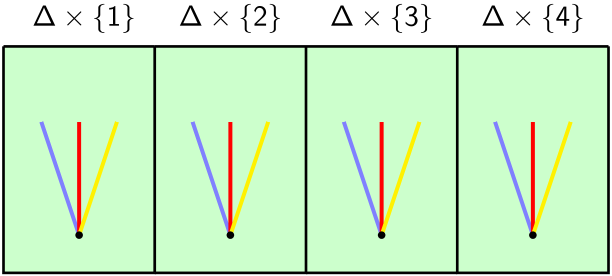

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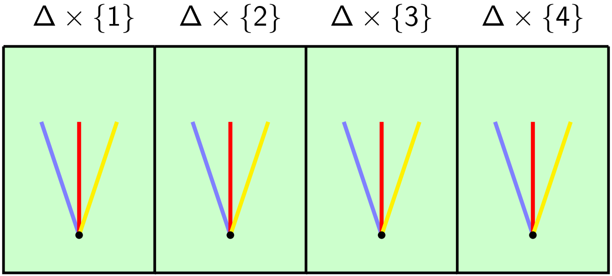

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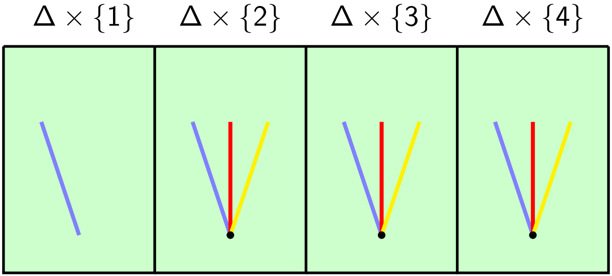

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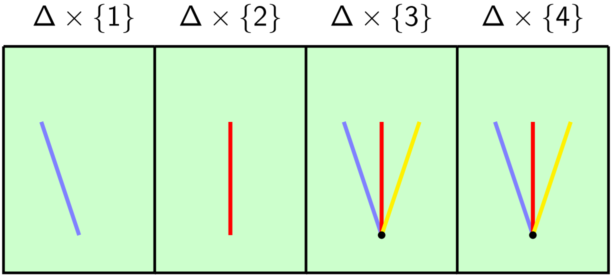

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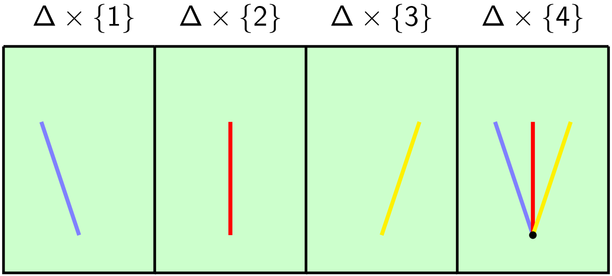

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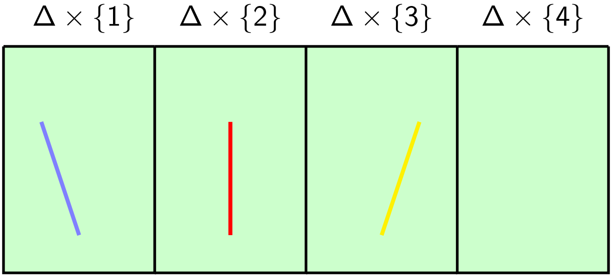

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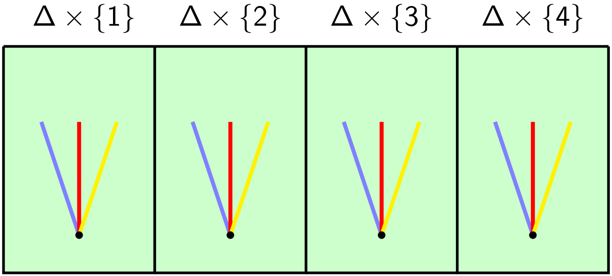

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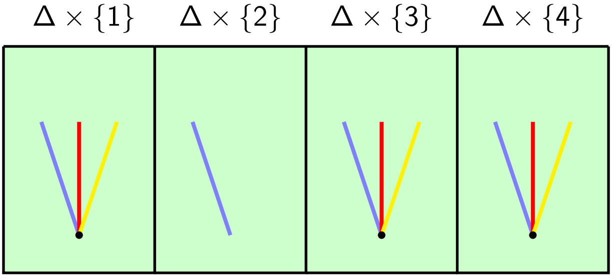

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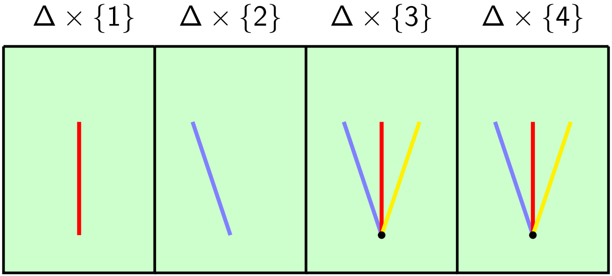

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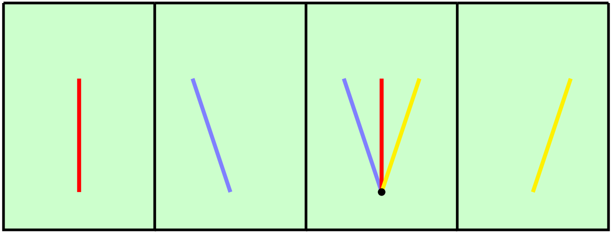

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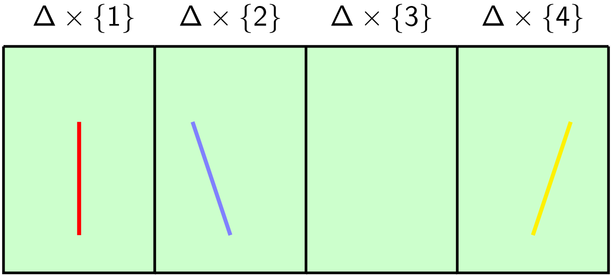

Examples arising from Cameron-Praeger construction

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Examples 3

1. the two 2-$(16, 6, 2)$ designs with $G \cong [2^4].S_4 < AGL_4(2)$ described by O'Relly-Reguerio (2006). Here, $\mathcal{D}_0 \cong AG_2(2)$.
2. the 2-$(45, 12, 3)$ with $G = [3^4].(10.4) < A\Gamma L_1(81)$ constructed by Praeger (2007). Here, $\mathcal{D}_0 \cong AG_2(3)$.
3. $\mathcal{D}$ the four 2-$(96, 20, 4)$ designs constructed by Law-Praeger-Reichard (2007). Here, $\mathcal{D}_0 \cong AG_2(4)$.
4. $\mathcal{D}$ is the 2-$(2^{2n}, 2^{n-1}(2^n - 1), 2^{n-1}(2^{n-1} - 1))$ design $S^-(n)$ with $n \geq 2$ described by Cameron-Seidel (1973), and $G \cong 2^{2n} : GL_2(n)$. Here, $\mathcal{D}_0 \cong AG_n(2)$.
5. $\mathcal{D}$ is a 2-$(1408, 336, 80)$ design constructed by Cameron-Praeger (2016) and $G \cong 2^6 : ((3 \cdot M_{22}) : 2)$. Here, $\mathcal{D}_0 \cong AG_3(4)$.

An Open Problem

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Does every flag-transitive, point-imprimitive symmetric 2-design arise from the Cameron-Praeger construction?

On the strenght of the Cameron-Praeger construction

Theorem [M., Praeger (2025+)]

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Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive **insoluble** automorphism group G preserving a partition Σ of $\mathcal{P}$ with v_1 classes of size v_0. If $\mathcal{D}_0$ is affine resolvable and $k_1 = v_1 - 1$, then one of the following holds:

1. $\mathcal{D}$ is one of the examples arising from the Cameron-Praeger contruction (showed above);
2. $\mathcal{D}$ is the 2-$(144, 66, 30)$ design constructed by Lempken (1999), and $G \cong M_{12}$.

On the strenght of the Cameron-Praeger construction

Theorem [M., Praeger (2025+)]

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive **insoluble** automorphism group G preserving a partition Σ of $\mathcal{P}$ with v_1 classes of size v_0. If $\mathcal{D}_0$ is affine resolvable and $k_1 = v_1 - 1$, then one of the following holds:

1. $\mathcal{D}$ is one of the examples arising from the Cameron-Praeger contruction (showed above);
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In order to establish whether the Lempken example arises or not from the CP-construction, we should check each $LS(10)$.

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Theorem [Alavi, Biliotti, Daneshkhah, M., Zhou et al. (2022)]

One of the following holds for $\left(\mathcal{D}_0, G_{\Delta}^{\Delta}\right)$:

(I) G_{Δ}^{Δ} **is almost simple** and one of the following holds:
  - (a) $\mathcal{D}_0$ is $AG_3(2)$ and $G_{\Delta}^{\Delta} = PSL_2(7)$;
  - (b) $\mathcal{D}_0$ is a 2-$(12,6,5)$ Witt design and $M_{11} \leq G_{\Delta}^{\Delta} \leq M_{12}$;

(II) G_{Δ}^{Δ} **is of affine type** and $\mathcal{D}_0$ is a 2-(p^i, p^j, λ_0) design with either $\lambda_0 = 1$ or $\lambda_0 = \frac{p^j-1}{p^{\gcd(j,i/z)}-1}$ for some $z \mid i$ such that $\gcd(j,z,i/z) = 1$, or $\lambda_0 = \frac{p^j-1}{a}$ for some $a \mid p^{\gcd(j,i)} - 1$.
The points and blocks of $\mathcal{D}_0$ are the points and (certain) j-subspaces of $AG_i(p)$.

Open Problems

- Affine resolvable 2-(v_0, k_0, λ_0) design $\mathcal{D}_0$
- Symmetric 2-(v_1, k_1, λ_1) design $\mathcal{D}_1$ with $k_1 = r_0 = v_1 - 1$
- $L = \{l_{ij}\}$ is a Latin square of order $r_0 + 1$

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3. Try to settle the G-soluble case.

Indice

Flag-transitive point-primitive 2-designs with small λ

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Theorem

Let $\mathcal{D}$ be a non-trivial 2-(v, k, λ) design admitting a flag-transitive point-primitive automorphism group G. If $G \not\leq A\Gamma L_1(v)$, v power of a prime, then $(\mathcal{D}, G)$ is classified in the following cases:

Conditions on $\mathcal{D}$	Conditions on G	Author(s)
$\lambda = 1$		Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck Saxl, 1990
$\lambda = 2, v = b$		O'Relly-Reguerio, 2005
$\lambda = 2, v < b$	G almost simple	Alavi, Devillers, Daneshkah, Liang, M., Praeger, Xia, Zhou et. al 2016–2025
$\lambda = 2, v < b$	G affine	Liang-M., 2025
$2 < \lambda \leq 10, v = b$	G affine	Alavi-Daneshkhah-M., 2025+

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Theorem (Zhang-Chen, 2023)

Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design with λ prime admitting a flag-transitive and point-primitive automorphism group G. Then the socle T of G is either nonabelian simple, or an elementary abelian p-group for some prime p.

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Theorem (Zhang-Chen-Zhou, 2024)

Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design with λ prime admitting a flag-transitive and point-primitive automorphism group G with socle $T \cong \mathrm{A}_n$, $n \geq 5$. Then one of the following holds:

1. $\mathcal{D}$ is a 2-$(6, 3, 2)$ design and $G \cong A_5$;
2. $\mathcal{D}$ is a 2-$(10, 4, 2)$ design and $G \cong A_5, S_5, A_6, P\Sigma L_2(9)$;
3. $\mathcal{D}$ is a 2-$(10, 6, 5)$ design and $G \cong A_5, S_5, A_6, S_6$;
4. $\mathcal{D}$ is a 2-$(15, 7, 3)$ design and $G \cong A_7, A_8$.

Sporadic Groups

Theorem (Alavi-Daneshkhah-M., 2025)

Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design with λ prime admitting a flag-transitive and point-primitive automorphism group G with socle T a simple sporadic group. Then $(\mathcal{D}, G)$ is (up to isomorphism) as one of the rows in the following table.

Table: Sporadic simple groups and flag-transitive 2-designs with λ prime.

Line	v	b	r	k	λ	G	G_α	G_B
1	12	22	11	6	5	M_{11}	$\mathrm{PSL}_2(11)$	A_6
2	22	77	21	6	5	M_{22}	$\mathrm{PSU}_3(4)$	$2^4{:}\mathrm{A}_6$
	22	77	21	6	5	$\mathrm{M}_{22}{:}2$	$\mathrm{PSU}_3(4){:}2$	$2^4{:}\mathrm{S}_6$
3	176	1100	50	8	2	HS	$\mathrm{PSU}_3(5){:}2$	S_8

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Example 4

Using the Higman-McLaughlin setting: $\cos(T, H, K) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where

- $\mathcal{P} = \{Hx : x \in T\}$, $\mathcal{B} = \{Ky : y \in T\}$;
- $Hx\ \mathcal{I}\ Ky$ if and only if $Hx \cap Ky \neq \varnothing$.

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Theorem (Alavi-Bayat-Daneshkhah-M., 2025)

Let $\mathcal{D}$ be a nontrivial symmetric design with λ prime admitting a flag-transitive and point-primitive automorphism group G of affine type. Then $G \leq A\Gamma L_1(q)$, or $\mathcal{D}$ is a symmetric 2-$(16, 6, 2)$ design with full automorphism group $2^4 : \mathrm{S}_6$ and point-stabilizer S_6.

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Example 5 (Buratti-Martinović-Nakić, 2025)

There are two non isomorphic flag-transitive 2-$(3^3, 6, 5)$ designs with $AGL_1(3^3) \trianglelefteq G \leq A\Gamma L_1(3^3)$.

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- Study the flag-transitive 2-(v, k, λ)-designs with $k \mid v$. Focus on the resolvable ones (joint work with E. Romano, Z. Lu, S. Zhou).
- Use the previous case to investigate the Cameron-Praeger construction in the non-symmetric design case.

THANK YOU FOR YOUR ATTENTION!