Splittings of toric ideals of graphs

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What is this talk about?

- To every finite, simple, connected graph G we will associate the toric ideal I_G, namely a prime ideal generated by binomials.
- Study when there are subgraphs G_1 and G_2 of G such that I_G can be written as $I_G = I_{G_1} + I_{G_2}$, where $I_{G_1} \neq I_G$ and $I_{G_2} \neq I_G$.

Definition of the toric ideal of a graph

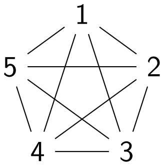
- Let G be a finite, simple, connected graph in the vertex set $V(G) = \{v_1, \ldots, v_n\}$ with the edge set $E(G) = \{e_1, \ldots, e_m\}$.
- Consider the polynomial rings $K[x_1, \ldots, x_m]$ and $K[t_1, \ldots, t_n]$ over a field K.
- Given an edge $e = \{v_i, v_j\}$ of G, we denote by $\mathbf{t^e}$ the monomial $t_i t_j$ of $K[t_1, \ldots, t_n]$.

Definition

The **toric ideal** I_G is the kernel of the K-algebra homomorphism $\phi : K[x_1, \ldots, x_m] \longrightarrow K[t_1, \ldots, t_n]$ given by $\phi(x_i) = \mathbf{t}^{e_i}$ for all $i = 1, \ldots, m$.

Binomials associated to even closed walks

Given an even closed walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of G, we write B_w for the binomial $B_w = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \in I_G$.



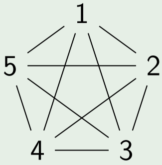
- $e_{ij} = \{i, j\} \rightsquigarrow x_{ij}$.
- $w_1 = (e_{12}, e_{23}, e_{34}, e_{14}) \rightsquigarrow B_{w_1} = x_{12}x_{34} - x_{23}x_{14} \in I_G$.
- $w_2 = (e_{12}, e_{23}, e_{13}, e_{14}, e_{45}, e_{15}) \rightsquigarrow B_{w_2} = x_{12}x_{13}x_{45} - x_{23}x_{14}x_{15} \in I_G$.

A generating set for I_G

Theorem (Villarreal 1995)

The ideal I_G is generated by all the binomials B_w, where w is an even closed walk of G.

Example



The set $S = \{x_{ij}x_{kl} - x_{il}x_{jk}, x_{ik}x_{jl} - x_{il}x_{jk} \mid 1 \leq i < j < k < l \leq 5\}$ is a minimal generating set for I_G.

Subgraph splittable toric ideals

Definition

A toric ideal I_G is called *subgraph splittable* if there exist subgraphs G_1 and G_2 of G such that $I_G = I_{G_1} + I_{G_2}$ and $I_{G_i} \neq I_G$ for each $i = 1, 2$. In this case, $I_G = I_{G_1} + I_{G_2}$ is called a *subgraph splitting*.

Question (Favacchio et al.)

For what graphs G can we find subgraphs G_1 and G_2 of G such that $I_G = I_{G_1} + I_{G_2}$ is a subgraph splitting?

Related papers

- G. Favacchio, J. Hofscheier, G. Keiper, A. Van Tuyl, *Splittings of toric ideals*, J. Algebra **574** (2021), 409–433.
- P. Gimenez, H. Srinivasan, *Gluing and splitting of homogeneous toric ideals*, J. Algebra **667** (2025), 911–930.

The graphs $G\backslash e$ and G_S^e

Notation

Given an edge e of G, we denote by $G\backslash e$ the graph with the same vertex set as G and whose edge set consists of all edges of G except e.

Definition (-, Thoma 2025)

Let $S = \{B_{w_1}, B_{w_2}, \ldots, B_{w_r}\}$ be a minimal generating set of I_G. Given an edge e of G, we define G_S^e to be the subgraph of G on the vertex set

$$V(G_S^e) = \bigcup_{1 \leq i \leq r \text{ and } e \in E(w_i)} V(w_i)$$

with edges

$$E(G_S^e) = \bigcup_{1 \leq i \leq r \text{ and } e \in E(w_i)} E(w_i).$$

Connection of $G \backslash e$ and G_S^e with our problem

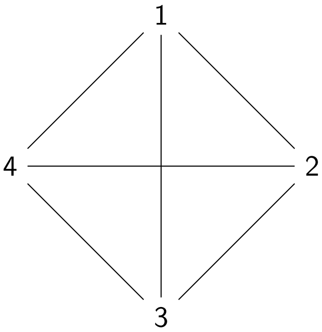
Theorem (-, Thoma 2025)

Let G be a graph and e be an edge of G. Let $S = \{B_{w_1}, B_{w_2}, \dots, B_{w_r}\}$ be a minimal binomial generating set of I_G with $r \geq 2$, then $I_G = I_{G_S^e} + I_{G \backslash e}$.

Question

Is $I_G = I_{G_S^e} + I_{G \backslash e}$ always a subgraph splitting of I_G?

Can it happen $I_{G_S^e} = I_G$?



- $w_1 = (e_{12}, e_{23}, e_{34}, e_{14}) \rightsquigarrow B_{w_1} = x_{12}x_{34} - x_{23}x_{14}$
- $w_2 = (e_{23}, e_{13}, e_{14}, e_{24}) \rightsquigarrow B_{w_2} = x_{23}x_{14} - x_{13}x_{24}$
- $S = \{B_{w_1}, B_{w_2}\}$ is a minimal generating set of I_G.
- Let $e = e_{23}$, then G_S^e is the whole graph G and therefore $I_G = I_{G_S^e} + I_{G \setminus e}$ is not a subgraph splitting of I_G.

When $I_G = I_{G_S^e} + I_{G\setminus e}$ is a subgraph splitting of I_G?

Theorem (-, Thoma 2025)

Let $S = \{B_{w_1}, \dots, B_{w_r}\}$ be a minimal binomial generating set of I_G with $r \geq 2$. Then

$$I_G = I_{G_S^e} + I_{G\setminus e}$$

is a subgraph splitting of I_G if and only if there is an edge $e \in E(w_i)$, $1 \leq i \leq r$, such that

$$I_{G_S^e} \neq I_G.$$

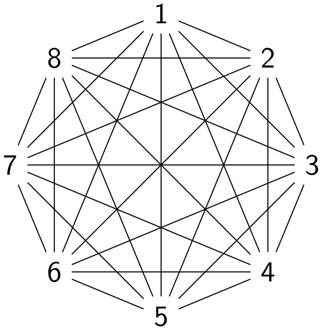
Classification of subgraph splittings

Theorem (-, Thoma 2025)

The toric ideal I_G is subgraph splittable if and only if there is an edge e of G and a minimal binomial generating set S of I_G such that $I_G = I_{G_S^e} + I_{G \setminus e}$ is a subgraph splitting of I_G.

Toric ideal of the complete graph

The *complete graph* K_n is the graph with n vertices in which each vertex is connected to every other vertex.

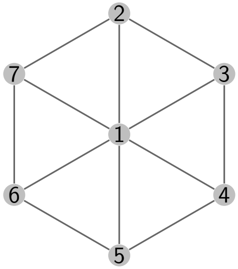


Theorem (-, Thoma 2025)

Let $n \geq 4$ be an integer and K_n be the complete graph on the vertex set $\{v_1, \ldots, v_n\}$. Then the toric ideal of K_n is subgraph splittable.

Wheel graph

Let W_n be the wheel graph with $n \geq 4$ vertices formed by connecting a single universal vertex to all vertices of a cycle of length $n-1$.

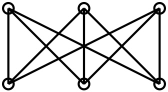


Theorem (-, Thoma 2025)

The toric ideal I_{W_n} is subgraph splittable if and only if $n = 5$ or n is even.

Complete bipartite graph

A bipartite graph G is called a *complete bipartite* graph if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge of V_1 is connected to every vertex of V_2. It is denoted by $K_{m,n}$, where m and n are the numbers of vertices in V_1 and V_2 respectively.



Theorem (-, Thoma 2025)

The toric ideal of $K_{m,n}$ is not subgraph splittable.