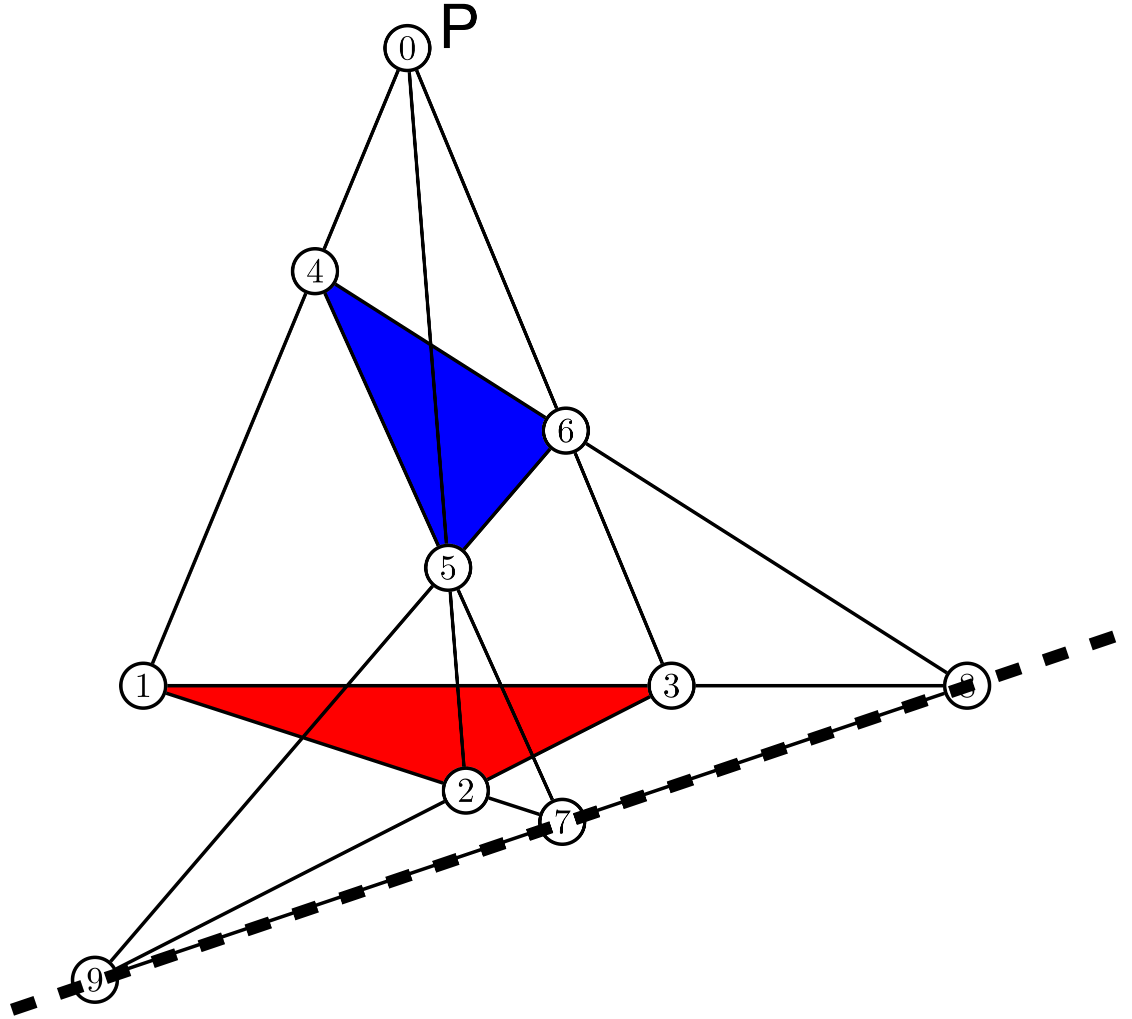
# Large Sets of Desargues Configurations

## 5th Pythagorean conference, Kalamata

**Anton Betten, Kuwait University, June 2025**

# Desargues' Theorem

*If two triangles are in perspective from a point, corresponding pairs of sides intersect in collinear points.*



Girard Desargues 1591-1661

# Incidence Geometry

- Forget about the coordinates and think of the incidence relation between points and lines:
- Elements are the 10 points,
- Sets are given by the elements incident with the lines.
- We have a set of ten 3-subsets of a 10-element set.
- Each set has size 3
- Each element lies in 3 sets
- Two distinct elements lie in at most one set.

0,1,4

0,2,5

0,3,6

4,6,8

1,3,8

5,6,9

2,3,9

4,5,7

1,2,7

7,8,9

# Configurations and Designs

- Such a set of sets is called a configuration.
- It is an example of a larger class of objects called designs.
- If we impose finiteness, we may call these objects *combinatorial designs*.
- A design may be related to a geometric object, but this is not necessary.
- We may encode a design by means of the *incidence matrix*: rows = points, columns = lines, entries which are 1 indicate incidence, the other entries are 0.

# The Incidence Matrix

0,1,4

0,2,5

0,3,6

4,6,8

1,3,8

5,6,9

2,3,9

4,5,7

1,2,7

7,8,9

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | | | | | | | |
| 1 | 1 | | | | 1 | | | | 1 | |
| 2 | | 1 | | | | | 1 | | 1 | |
| 3 | | | 1 | | 1 | | 1 | | | |
| 4 | 1 | | | 1 | | | | 1 | | |
| 5 | | 1 | | | | 1 | | 1 | | |
| 6 | | | 1 | 1 | | 1 | | | | |
| 7 | | | | | | | | 1 | 1 | 1 |
| 8 | | | | 1 | 1 | | | | | 1 |
| 9 | | | | | | 1 | 1 | | | 1 |

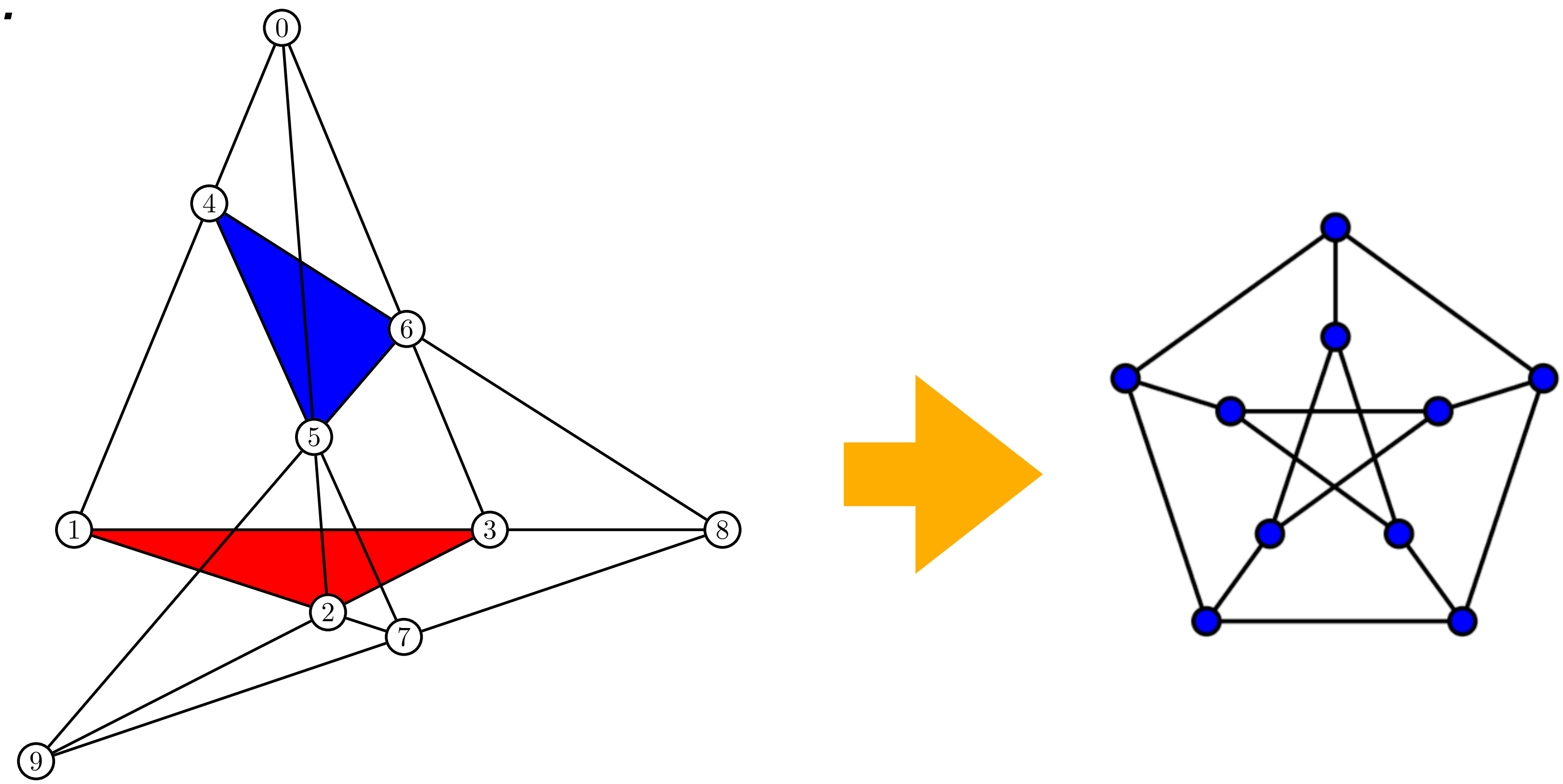
(empty spots are assumed to be zero)

# Isomorphism and Automorphism

- The nature of the points is irrelevant.
- If we relabel the points, and map the sets accordingly, the incidence structure maintains it's structure.
- We say that two incidence structures are *isomorphic* if there is a relabeling of the points that takes the sets of one to the sets of the other.
- The isomorphisms of one structure with itself are called *automorphisms*.
- They form a group, called the *automorphism group*.

# The Desargues Configuration $10_3$

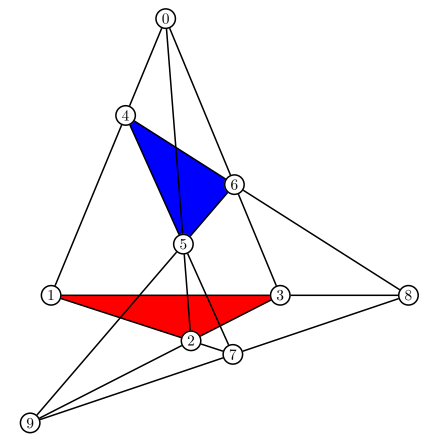
*The Desargues configuration has 120 automorphisms. The group is isomorphic to Sym(5), the symmetric group on 5 things.*




*The "not-collinear" relation on points induces the Petersen graph (with the same automorphism group).*

# Desargues’ Theorem: Relevance

Desargues’ Theorem holds in the Desarguesian projective planes, which are the planes arising from projective geometries.




In general projective planes, the theorem is not true (in general).

In fact, the classification of projective planes is based on the validity of Desargues’ Theorem (Lenz/Barlotti).

# Classification

- The Desargues configuration is unique.
- However, there are other configurations on 10 points with 10 lines, with 3 lines on each point and each line containing 3 points such that any two points are on at most one line:
- Coxeter 1954: Up to isomorphism, there are exactly ten configurations $10_3$.

# Subgroups

- The automorphism group comes with 4 classes of subgroups:
- The point stabilizer. Those automorphisms which stabilize a point.
- The line stabilizer: Those automorphisms which stabilize a line.
- The flag stabilizer: Those automorphisms which stabilize a flag (an incident point / line pair).
- The anti-flag stabilizer: Those automorphisms which stabilize an anti-flag (a non-incident point / line pair).
- For each type, there may be several classes.

# Orbits

- Let G be the automorphism group of an incidence structure $(\mathscr{V}, \mathscr{L}, \mathscr{R})$.

- The orbits of G on points, lines, flags, anti-flags (and possibly other things) matter.

- Each orbit comes with one associated conjugacy class of subgroups.

- The same subgroup may appear repeatedly.

- Under certain conditions, it is possible to recover the incidence structure from the subgroups.

# Synthetic Construction

- Let $G$ be a finite group. For a subgroup $H$, let $\tilde{H}$ be the conjugacy class of $H$.
- Let $A$,$B$,$C$ be subgroups, with classes $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, respectively.
- Define an incidence structure $(G, \tilde{A}, \tilde{B}, \tilde{C}) = (\mathscr{V}, \mathscr{L}, \mathscr{R})$ like so:
- $\mathscr{V} = \tilde{A}$,
- $\mathscr{L} = \tilde{B}$,
- $(V, L) \in \mathscr{R} \iff V \cap L \in \tilde{C}$.

# Synthetic Construction

- Theorem: $(G, \tilde{A}, \tilde{B}, \tilde{C}) = (\mathscr{V}, \mathscr{L}, \mathscr{R})$ is flag transitive iff the stabilizer of a point $A$, $G_A$, is transitive on pairs $(B, C) \in \tilde{B} \times \tilde{C}$ such that $A \cap B = C$.

- Example:

- Desargues = $(\mathrm{Sym}(5), \tilde{A}, \tilde{B}, \tilde{C})$ where $A = B$ is a subgroup of order 12 whose class has size 10 and $C$ is the elementary-abelian subgroup of order 4 whose class has size 15.

# Large Sets

- A large set “LS” of designs is a system $\mathscr{D}_i$ $i = 1, \ldots, N$ where all $\mathscr{D}_i$ is isomorphic designs with lines of size k on a set V and the union of the lines is disjoint and constitutes the set of k-subsets of V.
- Standard notation from design theory: v = size of the underlying point set, b = number of lines, k = size of each line.
- **Theorem**: If a LS of incidence structures $\mathscr{D}$ exists, then $b \mid \binom{v}{k}$.
- Next we wish to construct a large set of Desargues configurations.
- The large set is flag-transitive.

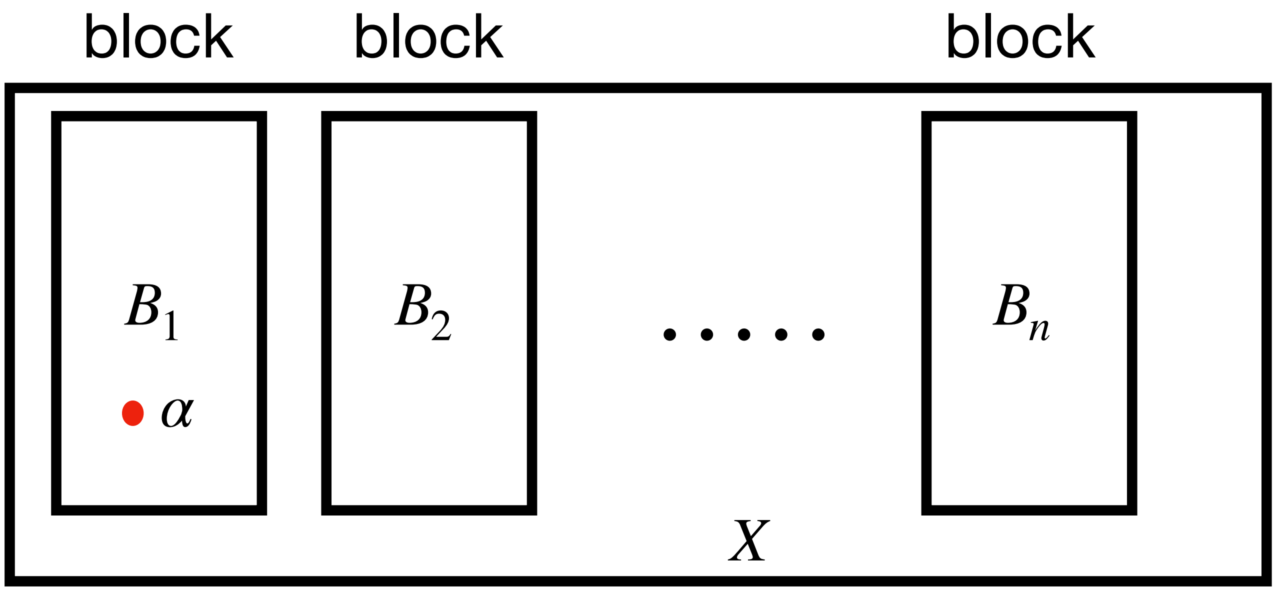
# Permutation Group Theory

- A transitive action by a group $G$ on a set $X$ is said to be imprimitive if there is a partition of $X$ into sets (called *blocks*): $X = B_1 \cup B_2 \cup \cdots \cup B_n$ such that $n > 1$ and $|B_i| > 1$.

- By a Theorem of Wielandt (1964), a transitive group $G$ acts primitively on $X$ if the stabilizer of a point, $G_\alpha$, is a maximal subgroup of $G$.

- For imprimitive groups, the possible block systems arise one-to-one from the overgroups of $G_\alpha$ (one block system arises from one overgroup).

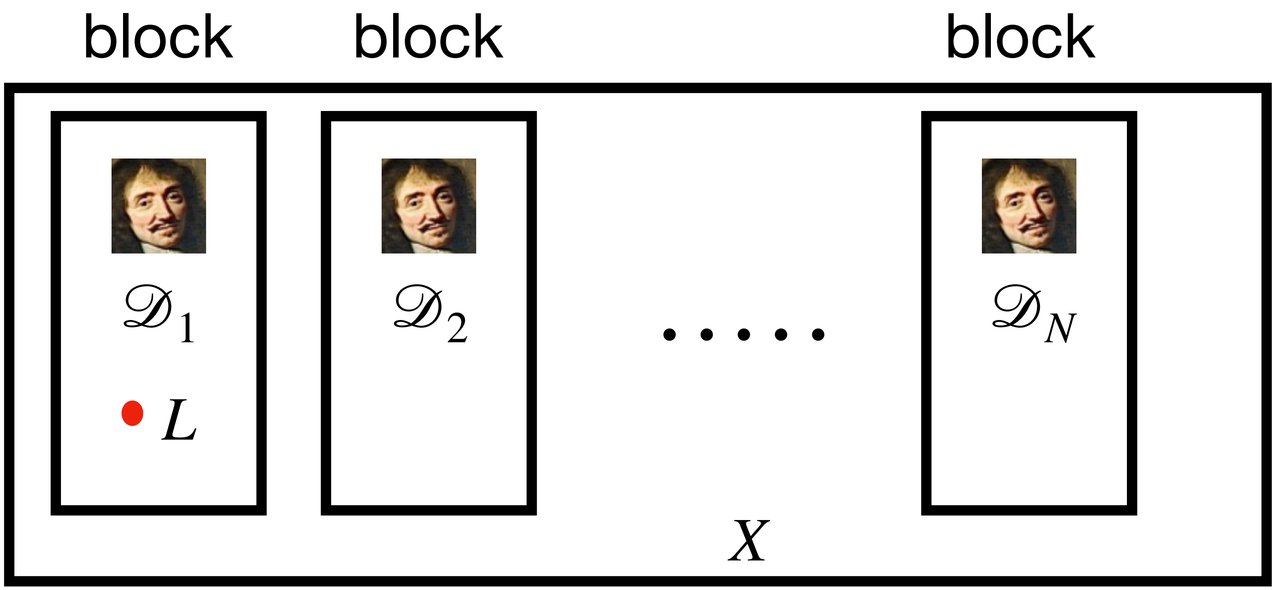
# Permutation Group Theory




$$G_\alpha < G_{B_1} < G$$

# Permutation Group Theory

Idea: use an imprimitive group to create a large set.




$$G_L < G_{\mathscr{D}_1} < G$$

Each $\mathscr{D}_i$ is a Desargues configuration of 10 lines. N=12.
L is a 3-subset of 10. X is the set of 3-subsets of 10.

# The Construction

- The large set has $N = \binom{10}{3}/10 = 12.$

- We need a group acting on 10 points that has a subgroup isomorphic to Sym(5).

- The group $\mathrm{P\Gamma L}(2,9) \simeq \mathrm{Aut}(\mathrm{Sym}(6))$ has this property. It has order 1440. It is related to the outer automorphism of Sym(6).

- The group $\mathrm{P\Gamma L}(2,9)$ is transitive on 3-subsets, so we can take L to be any 3-set we wish.

# The Construction

- We let $A$ be a point stabilizer (a self-normalizing subgroup of index 10)
- We let $B$ be a 3-set stabilizer (a self-normalizing subgroup of order 12).
- We let $\tilde{C}$ be a specific class of subgroups of order 4.
- We let $H$ be a subgroup isomorphic to Sym(5) which is a subgroup of Sym(6) which is inside Aut(Sym(6)).
- The orbit of H containing L gives one Desargues configuration. Why? (see below)
- The cosets of H in G give a total of 12 Desargues configurations.
- The result is a flag transitive large set of Desargues configurations with N=12.

Here is the flag-transitive large set of Desargues configurations.

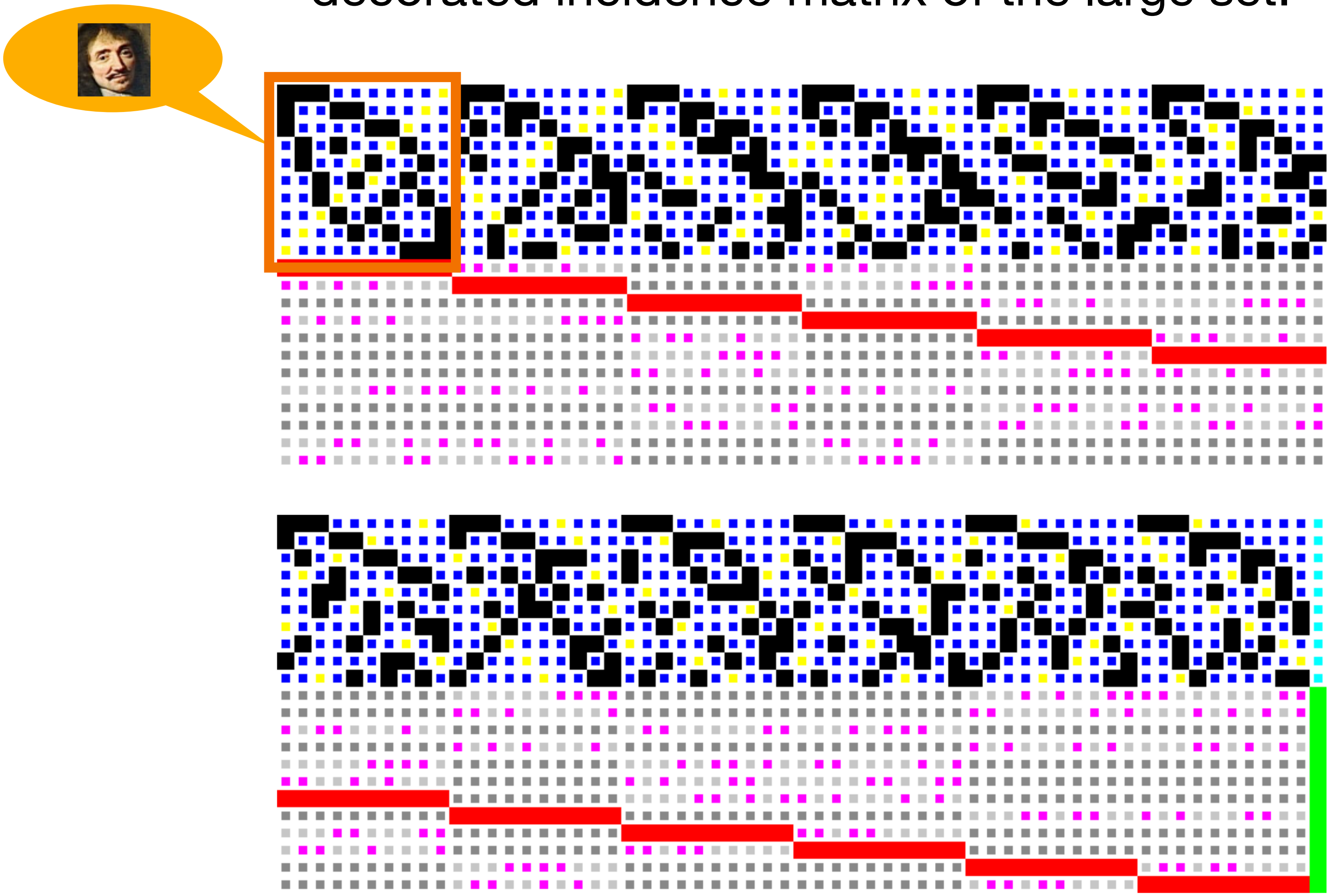
Each row is one Desargues configuration.

In total, each 3-subset of 10 appears exactly once:

- 012, 034, 056, 137, 158, 247, 268, 359, 469, 789
- 013, 024, 089, 127, 169, 259, 347, 368, 458, 567
- 014, 058, 069, 126, 138, 237, 249, 345, 579, 678
- 015, 026, 079, 128, 149, 239, 348, 367, 457, 568
- 016, 037, 049, 124, 157, 258, 269, 356, 389, 478
- 017, 028, 036, 156, 189, 234, 279, 357, 459, 468
- 018, 027, 045, 134, 179, 256, 289, 358, 369, 467
- 019, 038, 057, 136, 145, 235, 246, 278, 479, 689
- 023, 059, 068, 125, 139, 147, 248, 346, 578, 679
- 025, 039, 047, 123, 159, 168, 267, 378, 456, 489
- 029, 048, 067, 135, 146, 178, 236, 245, 379, 589
- 035, 046, 078, 129, 148, 167, 238, 257, 349, 569

you can check!

decorated incidence matrix of the large set:



the matrices are supposed to be side-by-side.

the colors represent flag-orbits and anti-flag orbits.

# The Construction

- Q: How do we know that one block of imprimitivity gives a Desargues configuration?
- A: Use the synthetic description of the Desargues configuration as a coset geometry. We have the same groups, just embedded as subgroup in some larger group.

# A pesky question

- Is this the only large set with these parameters?
- Or are there any others?

# Classification of Large Sets

- **Theorem**: There are, up to isomorphism, exactly 3617 large set of Desargues configurations. Exactly one is flag transitive, isomorphic to the example just presented. The other examples have smaller automorphism groups.

- The proof is based on exhaustive computer search.

| Aut | # of Isomorphism Types | Comment |
|---|---|---|
| 1 | 2178 | |
| 2 | 1111 | |
| 3 | 2 | |
| 4 | 244 | |
| 6 | 13 | |
| 8 | 59 | |
| 12 | 2 | |
| 16 | 9 | |
| 20 | 1 | |
| 24 | 2 | |
| 32 | 3 | |
| 48 | 1 | |
| 72 | 1 | |
| 1440 | 1 | flag transitive |

this is the example discussed previously.

# Comments

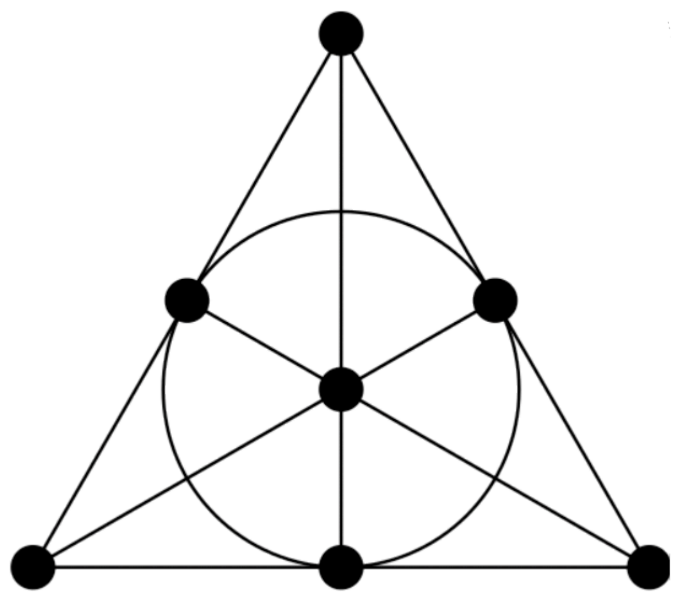
- In our research, we proceeded in the opposite direction.
- We started with the classification of large-sets of Desargues configurations (by computer).
- During the classification, one flag-transitive example appeared.
- The analysis of this example led to the description presented here.
- So, perhaps there is a lesson here:
- *The classification of a class of combinatorial objects is not the end but rather the beginning of something new.*

# Comments

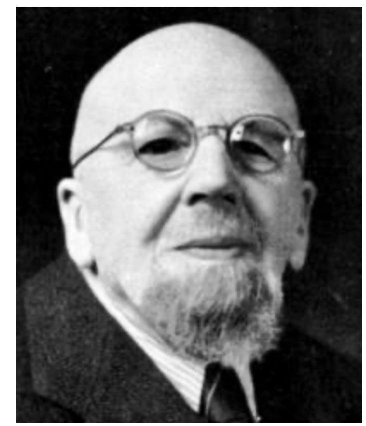
- There has been a lot of research on flag-transitive linear spaces (which are 2-designs):
  - Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck, Saxl (1990),
  - Pauley, Bamberg (2008),
  - Feng, Lu (2023).
- However, a configuration is not a linear space (it is only a 1-design), so the present work is different.

# Comments

- What about the Fano plane?



conf $7_3$ aka Fano plane

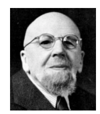


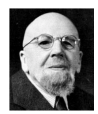
Gino Fano
1871-1952

# What about the Fano plane?

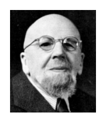
- There is no large set of Fano planes PG(2,2). Such a large set would have N=5 designs. However, N=2 is the largest size of a set of pairwise line-disjoint Fano-planes (N=2 can be done in 120 ways, not up to isomorphism).

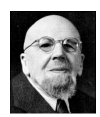
- Here is the best one can get:

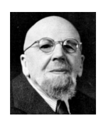
012,034,056,135,146,236,245 

013,025,046,126,145,234,356 

} 120 possibilities

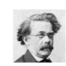
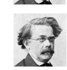
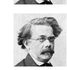
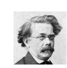
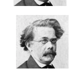
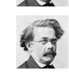
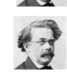
not possible 

not possible 

not possible 

# What about the Hesse configuration?

- There are *exactly two* large sets of Hesse configurations AG(2,3). These are configurations $9_4$ $12_3$. Here, we have 9 points and seven configurations, partitioning the 3-subsets of a 9 element set.
- Here is one (with automorphism group of order 42):

012, 034, 056, 078, 135, 147, 168, 238, 246, 257, 367, 458 
013, 024, 057, 068, 125, 148, 167, 236, 278, 347, 358, 456 
014, 023, 058, 067, 127, 138, 156, 245, 268, 346, 357, 478 
015, 026, 037, 048, 123, 146, 178, 247, 258, 345, 368, 567 
016, 025, 038, 047, 128, 137, 145, 234, 267, 356, 468, 578 
017, 028, 035, 046, 124, 136, 158, 237, 256, 348, 457, 678 
018, 027, 036, 045, 126, 134, 157, 235, 248, 378, 467, 568 

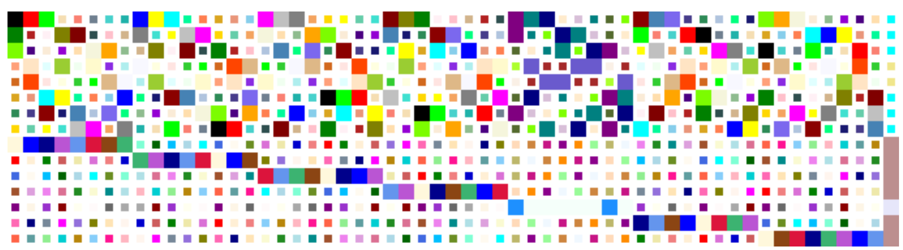
Otto Hesse
1811-1874

# The Configuration $8_3$

- There is a unique configuration $8_3$. It has an automorphism group of order 48. It arises from the affine plane AG(2,3) by removing one point.

- Up to isomorphism, there are exactly 3 large sets of configurations $8_3$. Two have an automorphism group of order 6. One has an automorphism group of order 21.

|Aut|=6

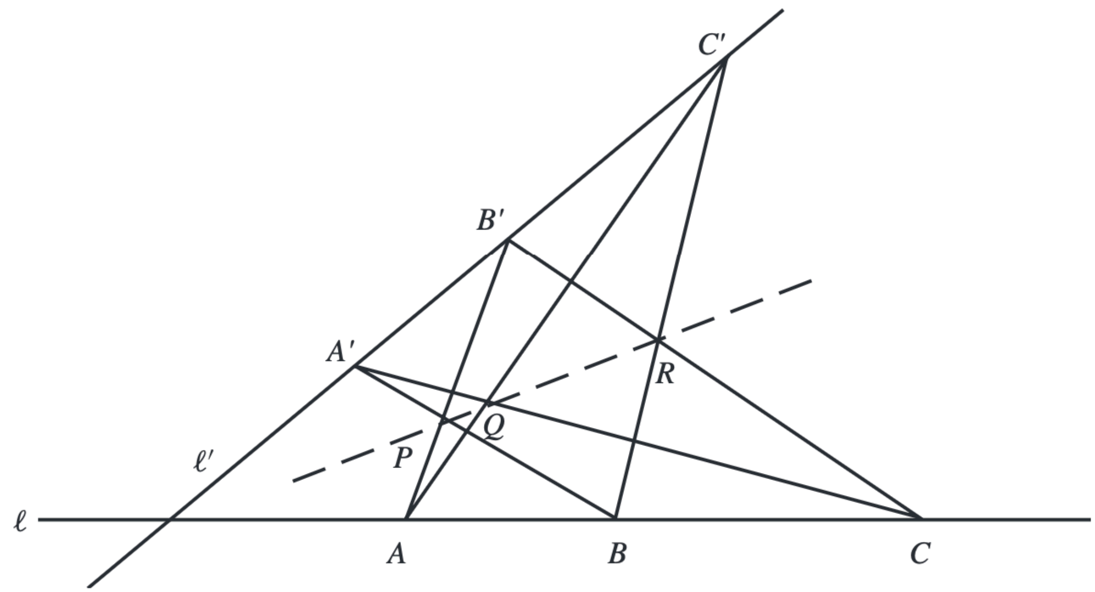


|Aut|=6

|Aut|=21

# The Configurations $9_3$

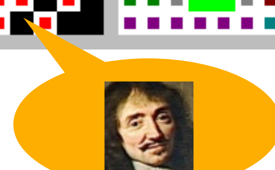
- There are exactly three configurations $9_3$. One is the Pappus configuration:




- However binomial(9,3) = 84 is not a multiple of 9, so there is no large set of $9_3$.

# Open Ends

- There are, up to isomorphism, ten $10_3$ configurations. One of them is the Desargues configuration. What about the other nine?



- The other nine have smaller automorphism groups. This makes classifying large sets computationally infeasible.

# Main Reference

- Anton Betten *Large sets of Desargues configurations*, under review.

Thank you for your attention!