Automorphism actions over nonabelian nilpotent coefficient groups, constructed via cohomology

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Hadamard matrices

Definition

A complex Hadamard matrix is a square $n \times n$ matrix H with entries $H_{i,j}$ of modulus 1, such that

$$HH^* = nI.$$

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- It is conjectured that real HM exist in all orders $n = 4m$.
- Complex HM exist in any order, e.g. the Fourier matrix.

Hadamard operations and equivalences

The space of all Hadamard matrix is stable under:

- Row permutations
- Column permutations
- multiplying by signs (phases) rows and columns.

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All this can be summarized as a transformation

$$H \mapsto LHR^*, \quad L, R \text{ are monomial}.$$

We define the automorphism group

$$\mathrm{Aut}(H) := \{(L, R) \text{ monomial} \mid LHR^* = H\}.$$

Noncommutative Hadamard matrices

Hadamard matrices can be defined over noncommutative domains, e.g. if μ is a group, we require

$$H \in \mu^{n\times n} \text{ satisfies } HH^* = nI,$$

in whatever ring $R \supset \mu$, and $H^*_{i,j} = H^{-1}_{j,i}$. For example $\mu =$the quaternion group.

Automorphisms are defined exactly the same way.

Remark

The notion of an automorphism is independet of the matrix being Hadamard. It applies to the whole space $\mu^{n\times n}$.

Motivation

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- The spaces in $R^{n \times n}$ cut by these symmetries are nice. They have structures of algebras or (bi-)modules.
- The search space becomes smaller if there is more symmetry. Orthogonality interacts ‘nicely’ with the space.
- This construction is suitable for other types of combinatorial and quantum objects, like weighing matrices, symmetric t-designs, ETFs, MUBs, SICs and more.

Outline of the talk

In this talk

- We will define the main problem: how to construct matrices with presecibed underlying group permutation action.
- We will set up a cohomological framework to interpret and solve this problem in the commutative case.
- We will see how to adapt this point of view to noncommuatative nilpotent coeffients.

$X \times Y$-matrices

The basic setting:

- Let G be a finite group, and X, Y be transitive G-sets.
- Let μ be the group of coeffients.
- We define the set of $X \times Y$-matrices:

$$M_{X \times Y} := \{f : X \times Y \to \mu\}.$$

- There is a natural action of G on $M_{X \times Y}$:

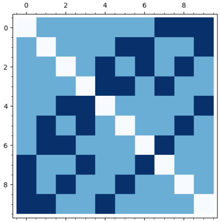
$$(gA)(x, y) := A(g^{-1}x, g^{-1}y).$$

Lemma

The subset $M^G_{X \times Y} := \{A | gA = A\}$ of G-invariants consists of matrices A with automorphism subgroup G.

Orbitals

The invariant space $M^G_{X\times Y}$ is the set of all matrices that are constant along the orbitals of $G \curvearrowright X \times Y$.



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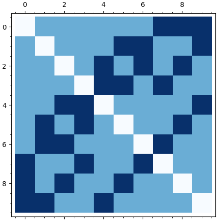

We are interested in the following problem:

Automorphism lifting problem

Find all $X \times Y$ matrices A, with automorphism subgroup G with the same underlying permutation action.

Our construction is related to

- Cocyclic development (de Launey and Horadam)
- Signed groups (R. Craygen)
- Representation theory: Centralizer algebras of monomial representations
- Weighted association schemes (D.G. Higman)

D-equivalence

Definition

We say that $X \times Y$-matrices A, B are D-equivalent, if

$$B = D_1 A D_2^*, \quad D_1, D_2 \text{ are diagonal over } \mu.$$

Write $A \sim_D B$.

Definition

We say that an $X \times Y$-matrix A is a cohomology-developed-matrix (CDM) if

$$\forall g \in G, \quad gA \sim_D A.$$

Cohomology-developed matrices are exactly what we are looking for.

Cohomology development

For the next few slides, we will assume that μ is commutative. Notice that $M_{X\times Y}$ is a group w.r.t. pointwise (Hadamard) multiplication, and $D := \{\text{rank 1 matrices} \in M_{X\times Y}\}$ a subgroup. We have

$$A \sim_D B \iff A \cong B \mod D.$$

Dividing by D:

the elements of the invariant group $(M_{X\times Y}/D)^G$
are CDMs up to D-equivalence.

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Dividing by D:

the elements of the invariant group $(M_{X\times Y}/D)^G = H^0(G; M_{X\times Y}/D)$. are CDMs up to D-equivalence.

Linear algebra approach

- We may view the conditions

$$\forall g \in G, \quad gA = LAR^*$$

  as a system of linear equations in the variables $A_{i,j}, L_{i,i}, R_{i,i}$.
- If e.g. $\mu = \{\pm 1\}$, this is a system over $\mathbb{F}_2$.
- This system clearly has to do with the group structure of G.
- Cohomology gives us more insight, and many tools that approach directly to the structures of G and μ.

Resolution and cohomology

We need to compute

$$H^0(G; M_{X\times Y}/D).$$

The module $M_{X\times Y}/D$ has the following resolution of G-modules:

$$0 \longrightarrow \mu \longrightarrow M_X \oplus M_Y \xrightarrow{\delta} M_{X\times Y} \longrightarrow M_{X\times Y}/D \longrightarrow 0 .$$

where $M_X = \{f : X \to \mu\}$ and the map $\delta(f,g)(x,y) := f(x)g(y)^{-1}$.

- This resolution implies that

$$H^0(G; M_{X\times Y}/D) \;\cong\; \mathbb{H}^0(0 \to \mu \cdots \to M_{X\times Y})$$

which leads to a spectral sequence.

Long exact sequences

More simply put, there are two long exact sequences to compute this cohomology:

$$\cdots H^0(G; M_{X\times Y}) \longrightarrow H^0(G; M_{X\times Y}/D) \longrightarrow H^1(G; (M_X \oplus M_Y)/\mu) \cdots$$

and

$$\cdots H^1(G; M_X \oplus M_Y) \longrightarrow H^1(G; (M_X \oplus M_Y)/\mu) \longrightarrow H^2(G; \mu) \cdots .$$

Thus our target group $H^0(G; M_{X\times Y}/D)$ is constructed by (subquotients) of $H^0(G; M_{X\times Y})$, $H^1(G; M_X \oplus M_Y)$ and $H^2(G; \mu)$.

The spectral sequence

The E^1-page of the spectral sequence is

$$
\begin{array}{ccccc}
H^2(G,\mu) & \xrightarrow{d_1^{2,-1}} & H^2(H_X,\mu)\oplus H^2(H_Y,\mu) & \xrightarrow{d_1^{2,0}} & \bigoplus_i H^2(T_i,\mu) \\
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\end{array} \quad . \qquad (1)
$$

Here H_X, H_Y and H_{O_i} are the stabilizer groups of X, Y and the orbitals O_i.

Theorem (AG and Giora Dula, 2023)

The set CDM of all $X \times Y$ CDMs is a group under the Hadamard product. It carries 3 step descending filtration, and the graded quotients are subquotients of $M^G_{X \times Y}$, $H^1(H_X; \mu) \oplus H^1(H_Y; \mu)$ and $H^2(G; \mu)$.

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- We can also cosider a G action on μ (e.g. for complex conjugation or Galois).
- It can be extended to cubes and higher tensors.
- It gives an alternative description of the Brauer group of a field.
- There is a relationship with equivariant cohomology.

Noncommutative coefficients

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If we restrict ourselves to nilpotent μ, linear algebra can be done in stages. This will also allow us to use cohomology.

In the following we will restrict our attention to nilpotent 2-step metabelian group μ. This covers the quaternions and the dihedrals.

For metabelian 2-step nilpotent μ, we have the extension sequence

$$1 \to Z \to \mu \to \overline{\mu} \to 1,$$

where Z is central and $\overline{\mu}$ is abelian, leading to the exact sequence of groups

$$1 \to M_Z \to M_\mu \to M_{\overline{\mu}} \to 1.$$

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- Write M_A/D for the D-equivalence quotient of M_A. We have maps of sets:

$$1 \to M_Z/D \to M_\mu/D \to M_{\overline{\mu}}/D \to 1.$$

These are no longer groups !

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- Consider the fiber $\pi^{-1}(\overline{A})$. If A_1, A_2 map to $\overline{A}$, then

$$A_2 = LA_1R^* \circ B, \quad B \in M_Z.$$

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  L, R diagonal.
- But B is not unique in M_Z/D (!)
- Let

$$C(\overline{A}) \ := \ \{[a, \overline{A}] \mid a \in \bar{\mu}\} \leq M_Z,$$

  ([,] is the commutator map)

Theorem (AG, 2027-)

There is an sequence of maps

$$1 \to (M_Z/D)^G \to (M_\mu/D)^G \xrightarrow{\pi} (M_{\overline{\mu}}/D)^G \xrightarrow{\delta} H^1(G; M_Z/D \cdot C(\overline{A})),$$

which is exact at $(M_{\overline{\mu}}/D)^G$ in the sense that $im(\pi) = \ker(\delta)$.

- This shows how we can lift CDMs from $\overline{\mu}$ to μ.
- For the obstruction term we have an exact sequence

$$H^1\left(G, \frac{M_Z}{D}\right) \to H^1\left(G, \frac{M_Z}{D \cdot C(\bar{A})}\right) \to H^2\left(G, \frac{C(\bar{A})}{D \cap C(\bar{A})}\right).$$

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- The group $H^1(G; M_Z/D)$ can be analyzed with the same resolution as for CDM, and involves $H^3(G; \mu)$.
- This construction generalizes step by step to all nilpotent μ.
- Solvable μ, general μ?

Questions?