A Reducibility Theorem for Minihypers

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1. Preliminaries

Definition. An $[n, k, d]_q$**-code** is a k-dimensional subspace of $\mathbb{F}_q^n$ where d is the minimal Hamming weight of a non-zero codeword.

Definition. (n, w)**-arc** in $\mathrm{PG}(r, q)$: a multiset $\mathcal{K}$ with

1) $|\mathcal{K}| = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n, w)**-blocking set (minihyper)** in $\mathrm{PG}(r, q)$: a multiset $\mathcal{F}$ with

1) $|\mathcal{F}| = n$;

2) for every hyperplane H: $\mathcal{F}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{F}(H_0) = w$.

Theorem 1. The following objects are equivalent:

(1) $[n, k, d]_q$ linear codes of full length and the maximal number of coordinate positions that are identical in all codewords is s;

(2) $(n, n-d)$-arcs in $\mathrm{PG}(k-1, q)$ with a maximal point multiplicity s;

(3) $(sv_k - n, sv_{k-1} - n + d)$-minihypers in $\mathrm{PG}(k-1, q)$.

Here: $v_k = \dfrac{q^k - 1}{q - 1}$.

Definition. An (n, w)-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called **t-extendable** if there exists an $(n + t, w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(r, q)$ with

$$\mathcal{K}'(P) \geq \mathcal{K}(P)$$

for all points P in $\mathrm{PG}(r, q)$.

Definition. An (n, w)-minihyper $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called **t-reducible** if there exists an $(n - t, w)$-arc $\mathcal{F}'$ in $\mathrm{PG}(r, q)$ with

$$\mathcal{F}'(P) \leq \mathcal{F}(P)$$

for all points P in $\mathrm{PG}(r, q)$.

Definition. An (n, w)-arc (or an (n, w)-minihyper) $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called **divisible** with divisor Δ if

$$\mathcal{K}(H) \equiv n \pmod{\Delta}$$

for every hyperplane H.

2. Classical Extension Results

Theorem 2. (R. Hill, P. Lizak, 1995) Let $\mathcal{K}$ be an (n, w)-arc in $\mathrm{PG}(r, q)$ with $w \equiv n + 1 \pmod{q}$. Assume that the multiplicities of all hyperplanes are congruent to n or $n+1$ modulo q. Then $\mathcal{K}$ can be extended to an $(n+1, w)$-arc.

Theorem 3. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, $w \equiv n-1 \pmod{q}$, such that the multiplicities of all hyperplanes are n or $n-1$ modulo q, then $\mathcal{F}$ can be reduced to an $(n-1, w)$-minihyper.

Theorem 4. (T. Maruta, 2001) Let $\mathcal{K}$ be an (n, w)-arc in $\mathrm{PG}(r, q)$, $q \geq 5$ odd, with $w \equiv n + 2 \pmod{q}$. Assume that the multiplicities of all hyperplanes are congruent to $n, n + 1$ or $n + 2$ modulo q. Then $\mathcal{K}$ can be doubly extendable to an $(n + 2, w)$-arc.

Theorem 5. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, $q \geq 5$ odd, with $w \equiv n - 2 \pmod{q}$, such that the multiplicities of all hyperplanes are $n, n - 1$ or $n - 2$ modulo q, then $\mathcal{F}$ can be reduced to an $(n - 2, w)$-minihyper.

Theorem 6. (Hitoshi Kanda, 2020) Let $\mathcal{K}$ be an (n, w)-arc in $\mathrm{PG}(r, 3)$ with $w \equiv n + 2 \pmod 9$ whose possible hyperplane multiplicities are all $n, n+1$, or $n + 2 \pmod 9$. Then $\mathcal{K}$ is doubly extendable to an $(n+2, w)$-arc.

Theorem 7. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, 3)$ with $w \equiv n - 2 \pmod 9$ whose possible hyperplane multiplicities are all $n, n-1$, or $n - 2 \pmod 9$. Then $\mathcal{F}$ is reducible to an $(n-2, w)$-minihyper.

3. The Main Theorem

Theorem 8. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, $q = p^h$, with $w \equiv n - q^j \pmod{q^{j+1}}$, $j \geq 0$. Assume that $\mathcal{F}$ has the following properties:

(1) $\mathcal{F}(H) \equiv n - q^j$ or $n \pmod{q^{j+1}}$ for every hyperplane H in $\mathrm{PG}(r, p)$;

(2) for every hyperplane H with $\mathcal{F}(H) \equiv n - q^j \pmod{q^{j+1}}$, $\mathcal{F}|_H = \mathcal{F}_1 + \chi_S$ for a unique $(j-1)$-dimensional subspace S and $\mathcal{F}_1$ is a divisible minihyper with divisor q^j;

(3) for every hyperplane H with $\mathcal{F}(H) \equiv n \pmod{q^{j+1}}$, $\mathcal{F}|_H$ is a divisible minihyper with divisor q^j.

Then $\mathcal{F} = \mathcal{F}' + \chi_T$, where $\mathcal{F}'$ is an $(n - v_{j+1}, w - v_j)$-minihyper, and T is an j-dimensional subspace. In addition, the subspace T is uniquely determined.

Corollary. Let $\mathcal{F}$ be an (n, w)-minihyper in $\mathrm{PG}(r, q)$, with $w \equiv n - q \pmod{q^2}$. Assume that $\mathcal{F}$ has the following properties:

(1) $\mathcal{F}(H) \equiv n - q$ or $n \pmod{q^2}$ for every hyperplane H in $\mathrm{PG}(r, q)$;

(2) for every hyperplane H with $\mathcal{F}(H) \equiv n - q \pmod{q^2}$, $\mathcal{F}|_H$ is reducible to a divisible minihyper with divisor q;

(3) for every hyperplane H with $\mathcal{F}(H) \equiv n \pmod{q^2}$, $\mathcal{F}|_H$ is a divisible minihyper with divisor q.

Then $\mathcal{F} = \mathcal{F}' + \chi_L$, where $\mathcal{F}'$ is an $(n - q - 1, w - 1)$-minihyper, and L is a line.

4. An Application

Theorem 9. A $(70, 22)$-minihyper in $\mathrm{PG}(4, 3)$ is one of the following:

(1) the sum of a solid and a $(30, 9)$-minihyper in $\mathrm{PG}(4, 3)$;

(2) the sum of a $(66, 21)$-minihyper in and a line in $\mathrm{PG}(4, 3)$.

Sketch of proof.

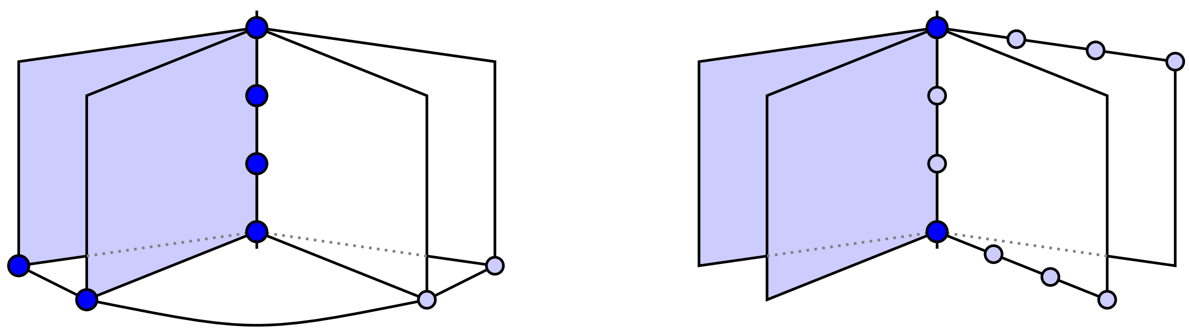
Step 1. Every $(70, 22)$-minihyper $\mathcal{F}$ is divisible by Ward's theorem, i.e. the possible hyperplane (solid) multiplicities are:

$$22, 25, 28, 31, 34, 37, 40, 43, 46, 49, \ldots$$

Step 2. If there exists a solid of multiplicity ≥ 49 then $\mathcal{F}$ is the sum of a $(40, 13)$-minihyper and a $(30, 9)$-minihyper

- $(40, 13)$ minihyper in $\mathrm{PG}(4, 3)$ – a solid;
- $(30, 9)$-minihyper in $\mathrm{PG}(4, 3)$ – either canonical or a $(30, 9)$-minihyper in $\mathrm{PG}(3, 3)$.

$(30, 9)$-minihypers in $\mathrm{PG}(3, 3)$



(a) (b)

(c) The complement of a cap in $\mathrm{PG}(3, 3)$

Step 3.

In case of $|\mathcal{F}|_H| < 49$:

- 28-solids are impossible;
- 37-solids are impossible;
- 46-solids are impossible.

The possible hyperplene multiplicities are: $22, 25, 31, 34, 40, 43.$

Step 4.

There exists no $(70, 22)$-minihyper in $\mathrm{PG}(4, 3)$ with a 22-hyperplane H such that $\mathcal{F}|_H$ is an irreducible $(22, 6)$-minihyper in $\mathrm{PG}(3, 3)$ and all solids have multiplicity ≤ 43.

Step 5.

Let $\mathcal{F}$ be a $(70, 22)$-minihyper with a maximal hyperplane multiplicity 43.

- If H is a hyperplane of multiplicity 22, then $\mathcal{F}|_H$ is reducible to a $(21, 6)$-minihyper;
- If H is a hyperplane of multiplicity 31, then $\mathcal{F}|_H$ is reducible to a $(30, 9)$-minihyper;
- If H is a hyperplane of multiplicity 40, then $\mathcal{F}|_H$ is reducible to a divisible $(39, 12)$-minihyper;

Step 6.

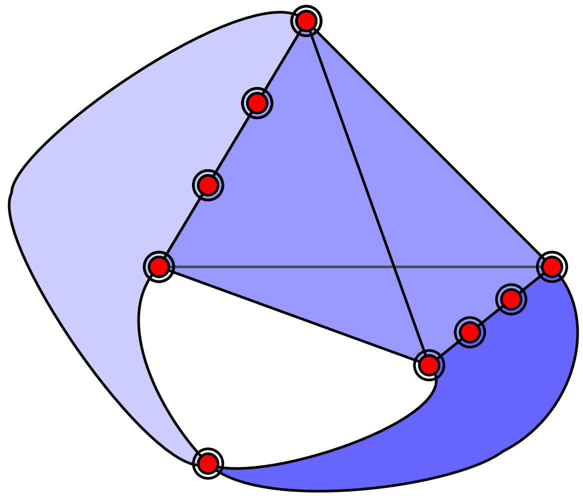
Let $\mathcal{F}$ be a $(70, 22)$-minihyper with a maximal hyperplane multiplicity 43.

- If H is a hyperplane of multiplicity 25, then $\mathcal{F}|_H$ is a divisible $(25, 7)$-minihyper;
- If H is a hyperplane of multiplicity 34, then $\mathcal{F}|_H$ is a divisible $(34, 10)$-minihyper;
- If H is a hyperplane of multiplicity 43, then $\mathcal{F}|_H$ is a divisible $(43, 13)$-minihyper;

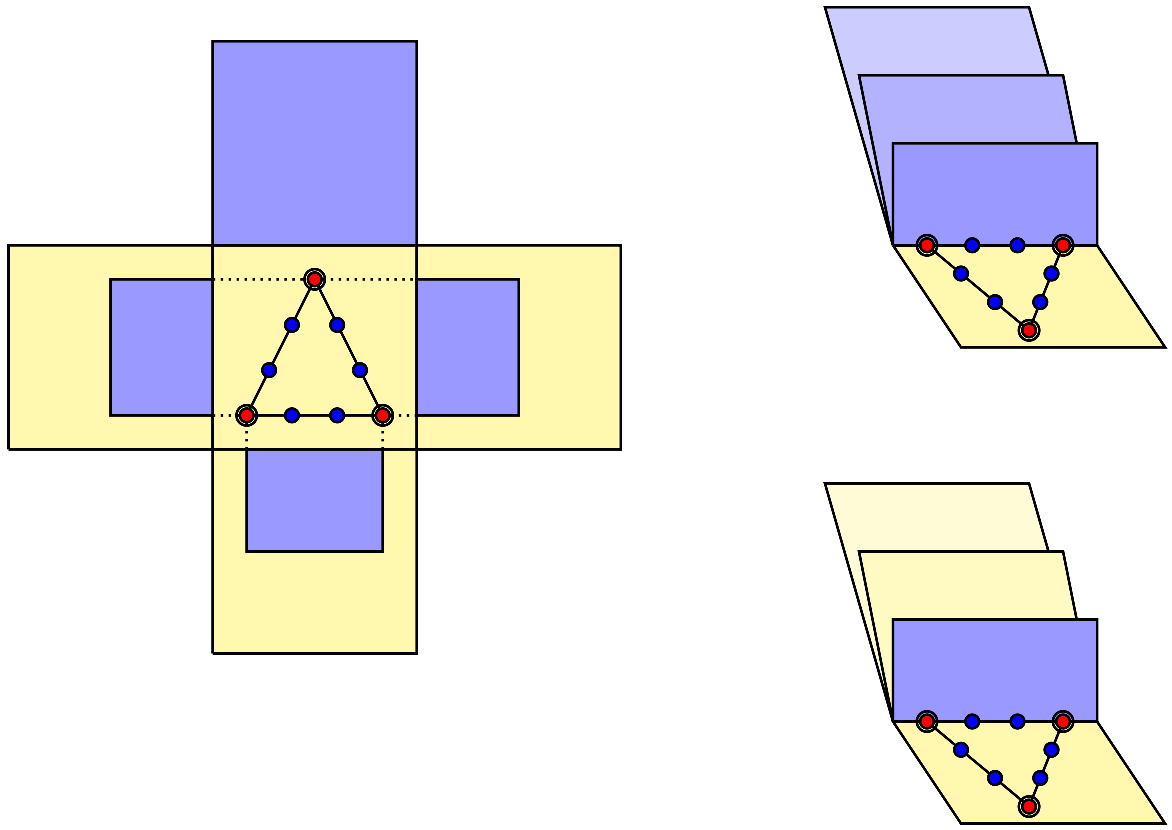
Hence the condition of Theorem 8 are satified and we get the minihyper from Theorem 9(2).

THANK YOU FOR YOUR ATTENTION!

(A) $(66, 21)$ **in** $\mathrm{PG}(4, 3)$



(B) $(66, 21)$ in $\mathrm{PG}(4, 3)$



(C) $(66, 21)$ in $\mathrm{PG}(4, 3)$