Combinatorial characterizations of ovoidal cones

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Classical problem in Projective Geometry

- Characterizing point sets of projective spaces by their intersection numbers with respect to suitable subspaces.
- Special attention has been given to those point sets that are arise as quadrics or Hermitean varieties.
- Also many other types of point sets have been considered.

Segre's theorem characterising irreducible conics

Theorem (Segre)

A set of $q+1$ points in $\mathrm{PG}(2, q)$, q odd, is an irreducible conic if it intersects each line in at most two points.

Ovoids and elliptic quadrics

An ovoid of $\mathrm{PG}(3, q)$ is a set X of $q^2 + 1$ with the property that through each of its points x there exists a unique plane π_x for which:

- every line through x contained in π_x intersects X in $\{x\}$;
- every line through x not contained in π_x intersects X in exactly two points.

Classical examples: elliptic quadrics (nonsingular quadrics of Witt index 1)

Theorem (Barlotti, Panella)

The ovoids of $\mathrm{PG}(3, q)$, q odd, are exactly the elliptic quadrics.

Characterising ovoids (and elliptic quadrics for q odd)

Theorem

A set of $q^2 + 1$ points of $\mathrm{PG}(3, q)$ intersecting each plane in either 1 or $q + 1$ points is an ovoid.

Theorem (Qvist)

A set of $q^2 + 1$ points of $\mathrm{PG}(3, q)$, $q \geq 3$, intersecting each line in at most two points is an ovoid.

A result of De Winter - Schillewaert (2010)

Theorem

Consider the projective space $\mathrm{PG}(n, q)$ with $n \geq 4$ and $q \geq 3$. Then any set of points having the same intersection numbers with respect to hyperplanes and co-dimension 2 subspaces as a nonsingular quadric or Hermitean variety living in $\mathrm{PG}(n, q)$ must be such a quadric or Hermitean variety.

Ovoidal cones

Consider in $\mathrm{PG}(4, q)$ a point p, a solid (3-dimensional subspace) Π not containing p and an ovoid O in Π. The union of all lines connecting p with the points of O is then called an ovoidal cone.

- Intersection numbers (or sizes) with respect to planes: 1, $q + 1$ and $2q + 1$;
- Intersection numbers with respect to solids: $q + 1$, $q^2 + 1$ and $q^2 + q + 1$.
- Innamorati and Zuanni (2021): Classification of all point sets having the same plane intersection numbers
- De Bruyn and Van de Voorde (2023): Classification of all point sets having the same solid intersection numbers

The result of Innamorati and Zuanni

Theorem

A set of points in $\mathrm{PG}(4, q)$ *such that every plane meets it in* $1, q+1$ *or* $2q+1$ *points is either:*

- *a parabolic quadric* $\mathcal{Q}(4, q)$*,*
- *an ovoidal cone,*
- *or the dual complete* 11*-cap in* $\mathrm{PG}(4, 3)$*.*

The dual complete 11-cap in $\mathrm{PG}(4,3)$

- Up to projective equivalence, there exists a unique cap X of 11 points in $\mathrm{PG}(4,3)$ such that any four of its points generate a solid containing exactly five points of X.
- Every solid intersects X in either 2 or 5 points.
- The set of 55 solids intersecting X in exactly 2 points forms a set of points of $\mathrm{PG}(4,3)^*$ intersecting each plane of $\mathrm{PG}(4,3)^*$ in either 1, 4 or 7 points of $\mathrm{PG}(4,3)^*$.

Characterisation result with respect to solid intersection numbers

- solid intersection numbers: $q+1$, q^2+1 and q^2+q+1.

Known examples which also block all planes:

- ovoidal cones
- planes (intersection numbers $q+1$ and q^2+q+1)

Main Theorem 1

Theorem (BDB & Van de Voorde)

Let $\mathcal{S}$ be a set of points of $\mathrm{PG}(4, q)$ which blocks all planes and has the same intersection numbers with respect to hyperplanes as an ovoidal cone. Then $\mathcal{S}$ is either a plane or an ovoidal cone.

What if we remove the blocking set condition?

An infinite family of examples

Theorem

Let Y be an ovoidal cone in $\mathrm{PG}(4, q)$ with vertex P. Let K be a line contained in Y, let Π_K be the unique solid intersecting Y in K and let L be a line in Π_K intersecting K in a single point distinct from P. The set $X := (Y \setminus K) \cup L$ then satisfies the following properties:

(1) *every solid of $\mathrm{PG}(4, q)$ intersects X in either $q + 1$, $q^2 + 1$ or $q^2 + q + 1$ points;*

(2) *there is a plane in $\mathrm{PG}(4, q)$ disjoint from X.*

Let us call these examples modified ovoidal cones.

Sporadic example 1

Theorem

Let π_1 *and* π_2 *be two planes in* $\mathrm{PG}(4,2)$ *intersecting in a single point* P*, and let* H *be a hyperoval of* π_2 *containing* P*. The set* $X := (\pi_1 \cup H) \setminus \{P\}$ *is then a set of* 9 *points of* $\mathrm{PG}(4,2)$ *satisfying the following properties:*

(1) *every solid of* $\mathrm{PG}(4,2)$ *intersects* X *in either* 3, 5 *or* 7 *points;*

(2) *there is a plane in* $\mathrm{PG}(4,2)$ *disjoint from* X*.*

Sporadic example 2

Take a frame $(P_1, P_2, P_3, P_4, P_5, P_6)$ in $\mathrm{PG}(4,2)$ and let Q_1, Q_2 and Q_3 be the respective third points on the lines P_1P_2, P_3P_4 and P_5P_6.

Theorem

The set $X := \{P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2, Q_3\}$ is a set of 9 points in $\mathrm{PG}(4,2)$ satisfying the following properties:

(1) *every solid of $\mathrm{PG}(4,2)$ intersects X in either 3, 5 or 7 points;*

(2) *there is a plane in $\mathrm{PG}(4,2)$ disjoint from X.*

Sporadic example 3

Take a frame $(P_1, P_2, P_3, P_4, P_5, P_6)$ in $\mathrm{PG}(4,2)$ and let Q_1, Q_2, Q_3, Q_4 and Q_5 be the respective third points on the lines P_1P_2, P_2P_3, P_3P_4, P_4P_5 and P_1P_5.

Theorem

The set $X := \{P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2, Q_3, Q_4, Q_5\}$ is a set of 11 points in $\mathrm{PG}(4,2)$ satisfying the following properties:

(1) *every solid of $\mathrm{PG}(4,2)$ intersects X in either 3, 5 or 7 points;*

(2) *there is a plane in $\mathrm{PG}(4,2)$ disjoint from X.*

Main Theorem 2

Theorem (BDB & Van de Voorde)

Let $\mathcal{S}$ be a set of points of $\mathrm{PG}(4, q)$ which does not block all planes and has the same intersection numbers with respect to hyperplanes as an ovoidal cone. Then $\mathcal{S}$ is either a modified ovoidal cone or one of the three sporadic examples described earlier.

Sketch of the proof

- Possible sizes for $\mathcal{S}$ are q^2+q+1, q^3+1, q^3+q+1 and q^3+q^2+1, but q^3+1 cannot occur if q is distinct from 2, 3 and 5 (by counting the three types of solids)
- Exclusion of the size q^3+1 for $q=3$ and $q=5$
- Two examples of size $q^3+1=9$ for $q=2$ (by computer)
- Sizes still need to be treated: q^2+q+1, q^3+q+1 and q^3+q^2+1
- For each of these, we can determine how many solids there are of each type
- Examples of size q^2+q+1 are planes (easy, a few lines)
- Exclusion of the size q^3+q^2+1 (1 page)
- Treatment of the size q^3+q+1 (13 pages)

The case $|\mathcal{S}| = q^3 + q + 1$

- $q = 2$: by computer (two examples)
- Suppose $q \geq 3$.
- Focus on the $q^2 + 1$ solids that meet $\mathcal{S}$ in exactly $q + 1$ points.
- The following was essential in our treatment.

Lemma

If Π is a solid intersecting $\mathcal{S}$ in exactly $q + 1$ points, then $\Pi \cap \mathcal{S}$ is either a line or a set of the form $(L \setminus \{P\}) \cup \{Q\}$, where L is a line, P is a point of L and Q is some point outside of L.

To show that a modified ovoidal cone arises, we needed to show that the point P is always the same point if the second possibility occurs (= vertex of ovoidal cone).

Proof of the essential lemma

- For $q > 5$, this relied on a result of Dodunekov, Storme and Van de Voorde (see below, for $a = 1$ and $n = 3$).
- For $q \in \{3, 4, 5\}$: hard work (most difficult part in the proof).

Lemma

If $\mathcal{S}$ is a partial $(q + a)$-cover of $\mathrm{PG}(n, q)$, $a < \frac{q-2}{3}$, with at most q^{n-1} holes, then there are at least $q^{n-1} - aq^{n-2}$ holes and the holes are contained in one hyperplane.

Dual version for $a = 1$ and $n = 3$, and consequence

Lemma

Suppose X is a set of $q+1$ points of $\mathrm{PG}(3,q)$, $q > 5$, which is not a plane blocking set for which there are at most q^2 disjoint planes. Then there are at least $q^2 - q$ planes in $\mathrm{PG}(3,q)$ disjoint from X and all these planes go through a certain point P.

- But then $X \cup \{P\}$ is then a plane blocking set of size $q+2$ in $\mathrm{PG}(3,q)$.
- A result of Beutelspacher then implies that $X \cup \{P\}$ is a line L, plus some point Q.
- This implies that $X = (L \setminus \{P\}) \cup \{Q\}$.