Codes and Designs in Polar Spaces

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University of Amsterdam

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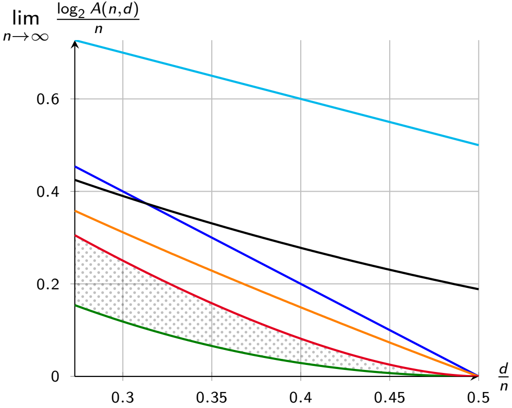

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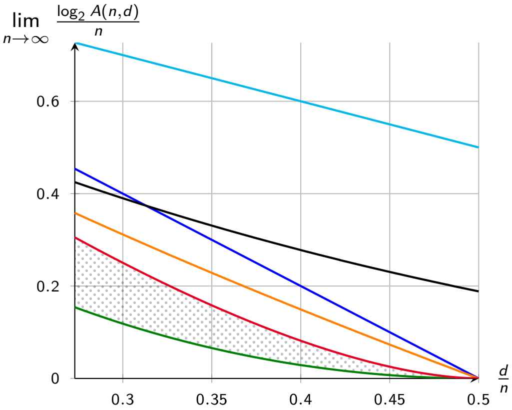

All these upper bounds come from a linear program whose optimal solution is unknown.

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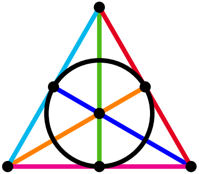

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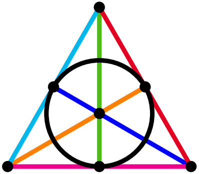

A *t-design* is a collection Y of n-subsets of a v-set V such that each t-subset of V lies in exactly λ members of Y.

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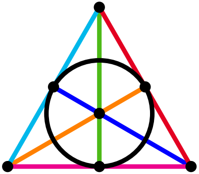

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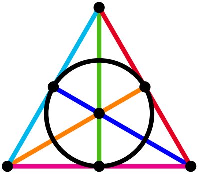

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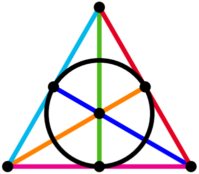

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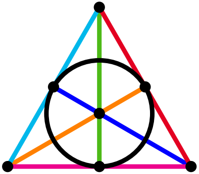

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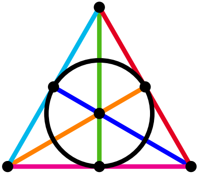

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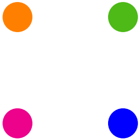
Keevash 2014, Glock-Kühn-Lo-Osthus 2016

A t-Steiner system exists if v is large enough and some natural divisibility conditions are satisfied.

Extremal combinatorics



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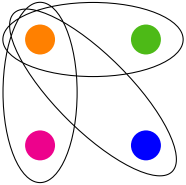

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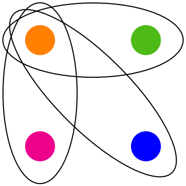

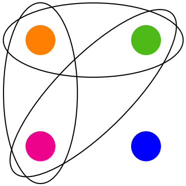

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For $v \geq 2n$, the size of an intersecting family of n-subsets of a v-set is at most $\binom{v-1}{n-1}$.

Wilson 1984

For v sufficiently large compared to t, the size of a t-intersecting family of n-subsets of a v-set is at most $\binom{v-t}{n-t}$.

q-analog problems

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n-tuples over $\{0, 1\}$

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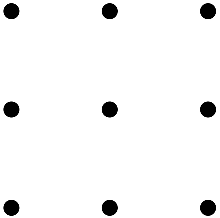

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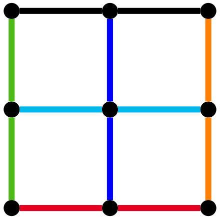

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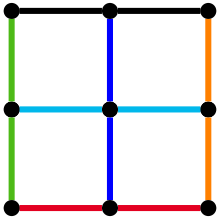

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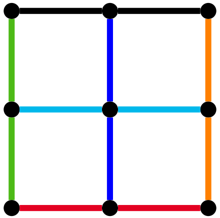

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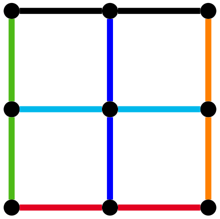

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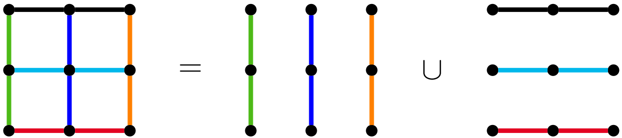
The maximal subspaces have the same dimension, called the rank of the polar space.

The six families of polar spaces

Up to isomorphism, there are six polar spaces of rank n.

form	name	type
Hermitian	Hermitian	${}^2A_{2n-1}$
Hermitian	Hermitian	${}^2A_{2n}$
alternating	symplectic	C_n
quadratic	hyperbolic	D_n
quadratic	parabolic	B_n
quadratic	elliptic	${}^2D_{n+1}$

Bipartite halves of D_n



The hyperbolic polar space D_2

Bipartite halves $\frac{1}{2}D_2$

Subspace codes

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Every d-code Y in $\mathbb{F}_q^v$ satisfies

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	$\mathbb{F}_q^v$	${}^2A_{2n-1}$	$\frac{1}{2}D_m$
b	q	$-q$	q^2
c	q^{v-2n}	-1	$1/q$ or q

Bounds

Let X be the set of n-spaces in $\mathbb{F}_q^v$, ${}^2A_{2n-1}$, or $\frac{1}{2}D_m$.

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Corollary (Schmidt-W. 2023)

The bounds in ${}^2A_{2n-1}$ and $\frac{1}{2}D_m$ imply bounds for codes in all other polar spaces.

Are the bounds sharp?

Theorem (Schmidt-W. 2023)

The bounds are sharp up to a constant factor in the

- Hermitian polar space ${}^2A_{2n-1}$ for odd d,
- symplectic polar space C_n for odd d,
- parabolic polar space B_n for odd d and even q,
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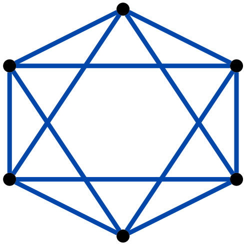
Remaining bounds are met up to a small power of q^n.

A distance-regular graph



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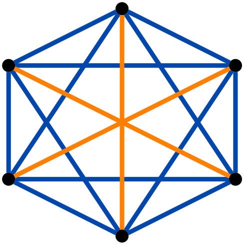

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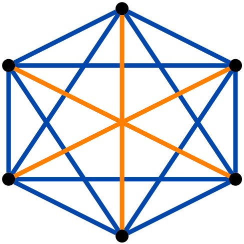

$$A_1 = \begin{pmatrix} J - I & J - I \\ J - I & J - I \end{pmatrix}$$

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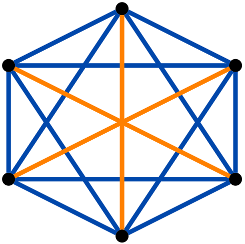

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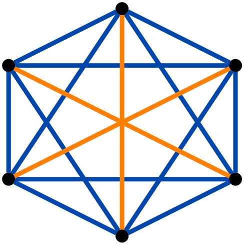

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$$= \begin{cases} 4 & \text{if } x = y \\ 2 & \text{if } (A_1)_{x,y} = 1 \\ 4 & \text{if } (A_2)_{x,y} = 1 \end{cases}$$

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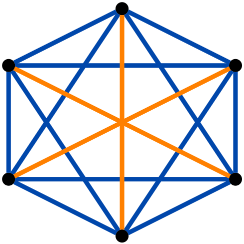

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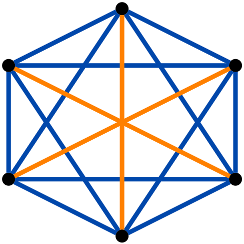

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$$A_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

The matrices I, A_1, A_2 generate a commutative algebra:

$$\begin{aligned} A_1 A_2 &= A_1 \\ A_1^2 &= 4I + 2A_1 + 4A_2 \\ A_2^2 &= I. \end{aligned}$$

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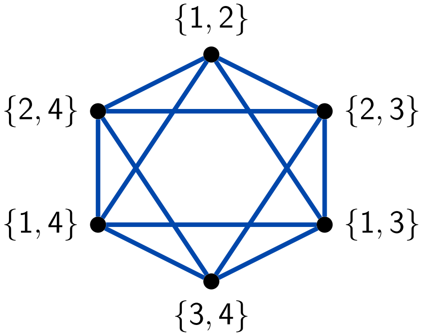
This algebra is called the Bose-Mesner algebra.

Classical examples

Johnson graph $J(n, v)$

for $(n, v) = (2, 4)$



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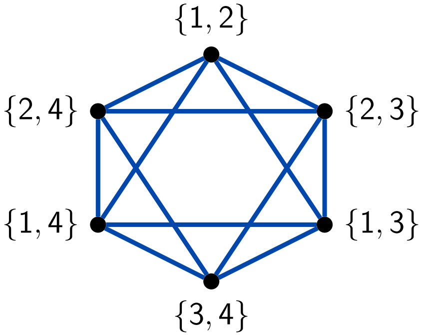

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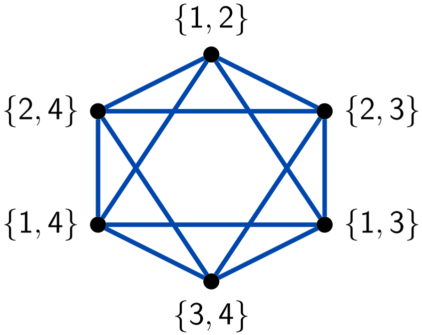

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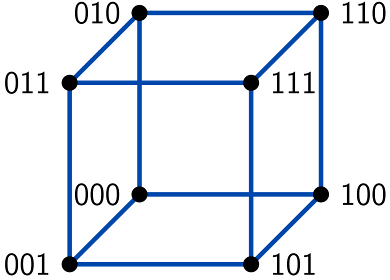

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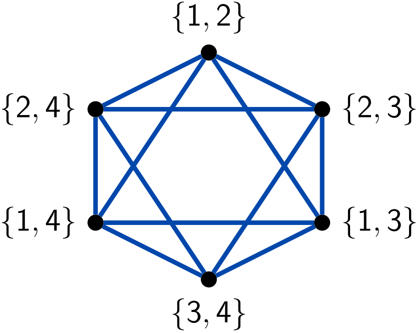

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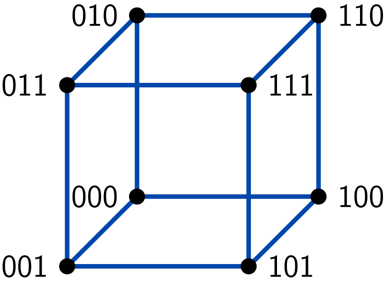

$$\rho(x, y) = \#\{i \mid x_i \neq y_i\}$$

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So, $Q_k(i)$ corresponds to V_k.

Polynomial structures

Every distance-regular graph is P-polynomial, that is

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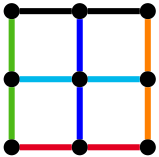

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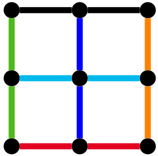

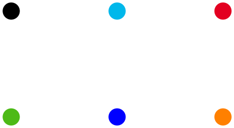

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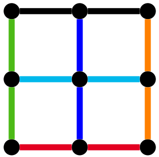

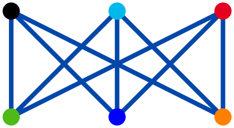

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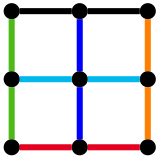

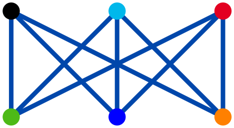

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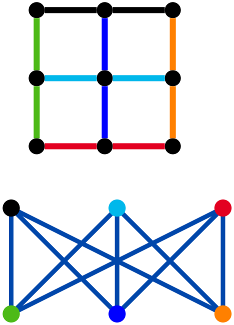

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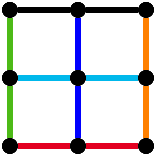

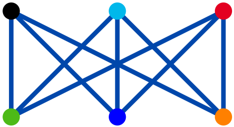

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How can we derive upper bounds on such codes?

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Numerical comparison to the LP optimum shows:

In most cases, the bound $(*)$ is not optimal!

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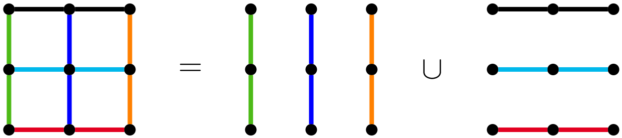

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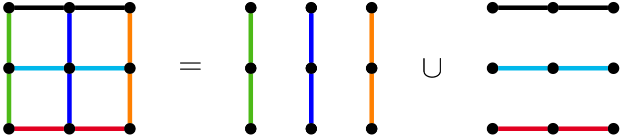

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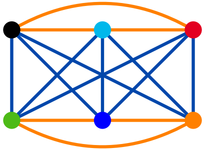

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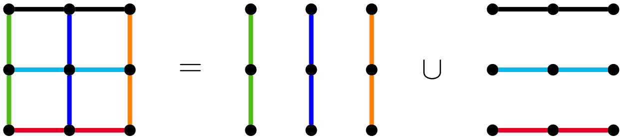

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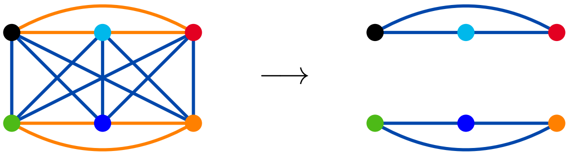

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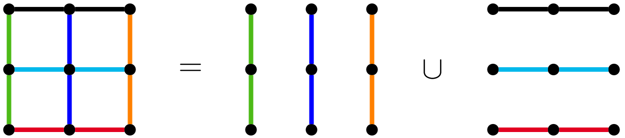

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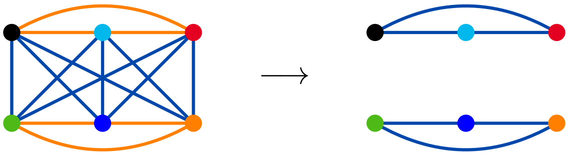

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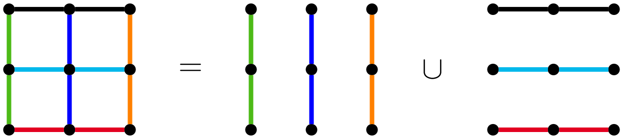

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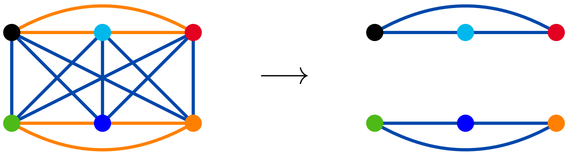

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Q-numbers of $\mathbb{F}_q^v$, ${}^2A_{2n-1}$, and $\frac{1}{2}D_m$

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Delsarte's LP with $\prod_{i=d}^{n}(z - z_i)$ and the q-Hahn polynomials instead of the q-Krawtchouk polynomials gives our bounds.

Linear programming optimum

Theorem (Schmidt-W. 2023)

Our bound for d-codes in a polar space is precisely the optimum of Delsarte's linear program for the

- Hermitian polar space ${}^2A_{2n-1}$,
- parabolic polar space B_n for odd d,
- symplectic polar space C_n for odd d,
- hyperbolic polar space D_n for even d.

Intersecting sets

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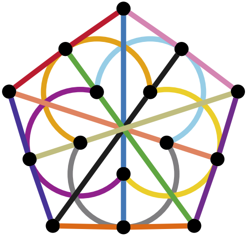

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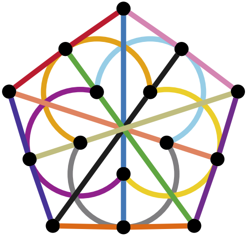

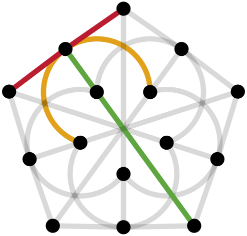

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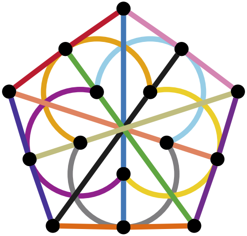

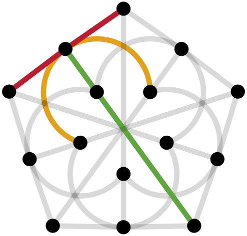

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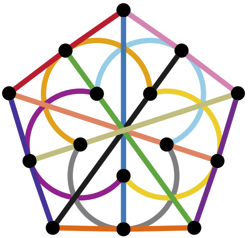

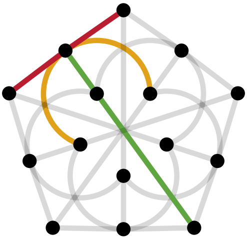

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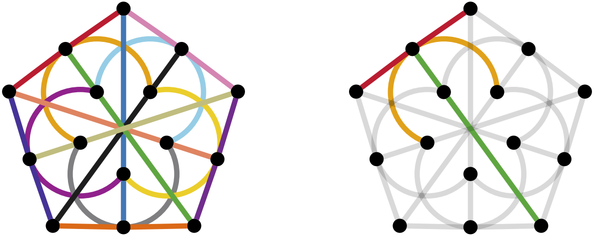

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How large can a t-intersecting set be?

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1-intersecting

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Upper bounds via Hoffman bound.

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t-intersecting with $t > 1$

- Ihringer-Metsch (2018)

Classification of largest t-intersecting sets for $n \lessapprox 3t$. Also upper bounds via Hoffman bound for all q, n, t.

Erdős-Ko-Rado-type bounds

Corollary (Schmidt-W. 2025+)

A t-intersecting set Y with $1 < t < n$ satisfies

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Erdős-Ko-Rado-type bounds

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They improve the bounds from Ihringer-Metsch (2018), but are still far away from the largest known examples.

Proof

A t-intersecting set Y in a polar space of rank n satisfies

$$\rho(x, y) = n - \dim(x \cap y) \leq n - t$$

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$\mathrm{LP}(n - t + 1)$ is the LP optimum for $(n - t + 1)$-codes.

Designs over $\mathbb{F}_q$

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Both results use probabilistic methods.

Steiner systems in polar spaces

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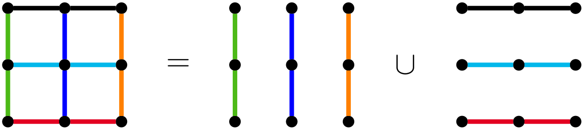

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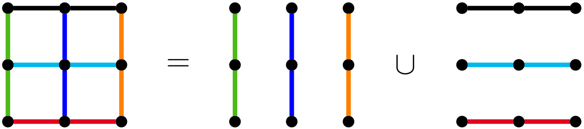

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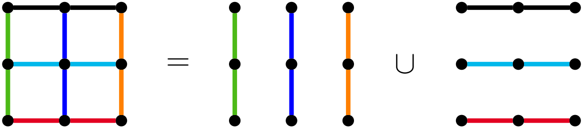

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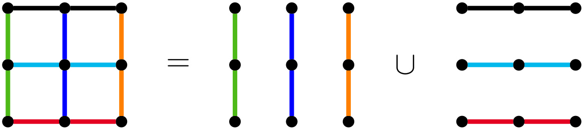

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1-Steiner systems are *spreads* in polar spaces.

Except for $\frac{1}{2}D_n$ and spreads in some polar spaces, no other nontrivial Steiner systems are known.

Classification of Steiner systems

Theorem (Schmidt-W. 2023)

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Conjecture

$\frac{1}{2}D_n$ are the only nontrivial t-Steiner systems with $t > 1$.

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Do t-designs in polar spaces exist for all t?

Known examples

De Bruyn-Vanhove (2012), Bamberg-Lansdown-Lee (2018)
There are 2-designs in the parabolic polar space B_3 for $q = 3, 5, 7, 11$. There exists a 2-design in the elliptic polar space ${}^2D_4$ for $q = 2$.

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Existence for all $t \geq 3$?

Existence of designs

Theorem (W. 2025)

Let $\mathcal{P}$ be a polar space of rank n. For all positive integers t and k with $k > 10.5\,t$ and for n large enough with $n > k^2$, there exists a t-(n, k, λ) design in $\mathcal{P}$ whose size is at most q^{21nt}.

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The proof is nonconstructive, based on a probabilistic method (by Kuperberg-Lovett-Peled, 2017).

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spaces

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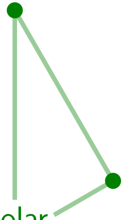

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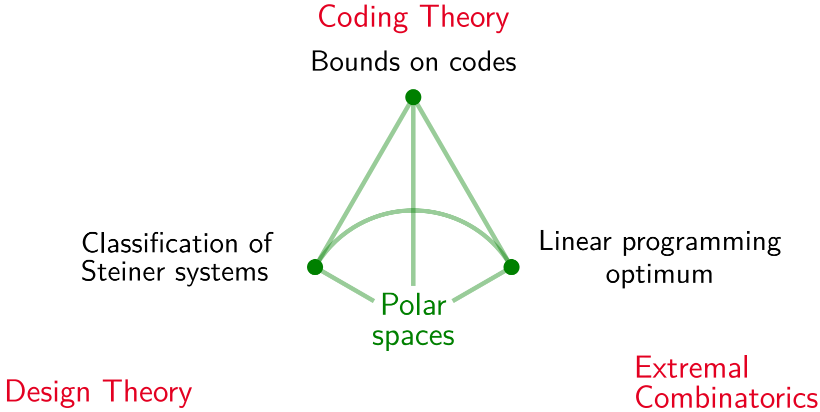
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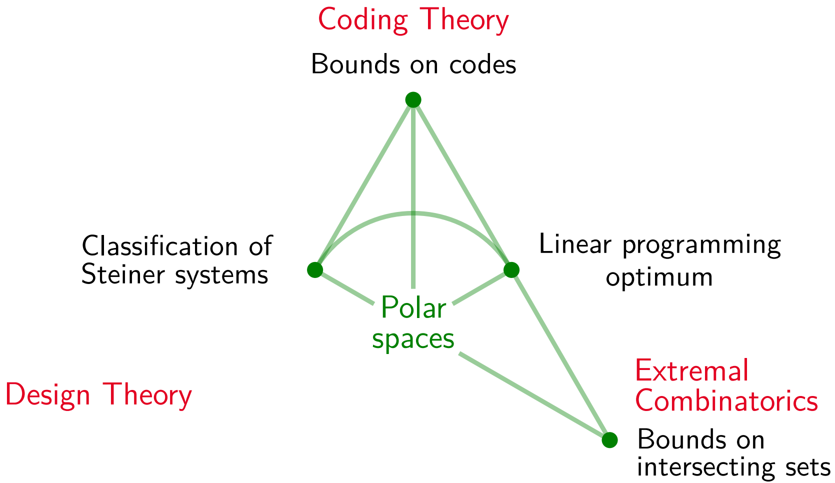

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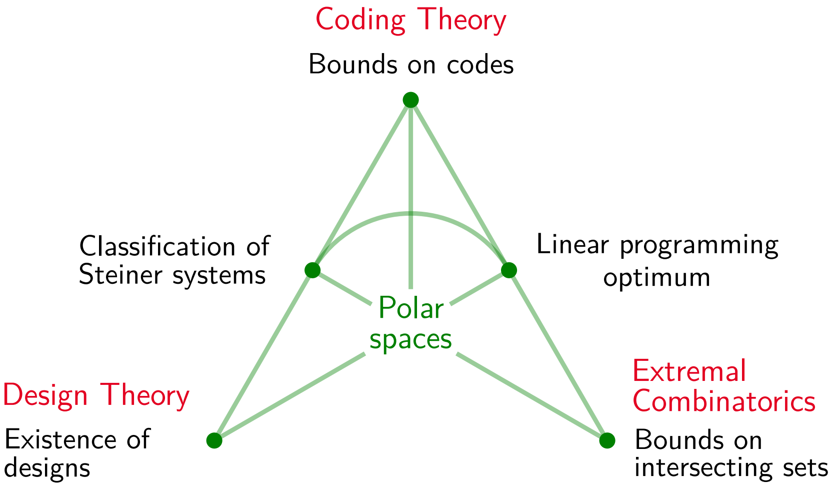

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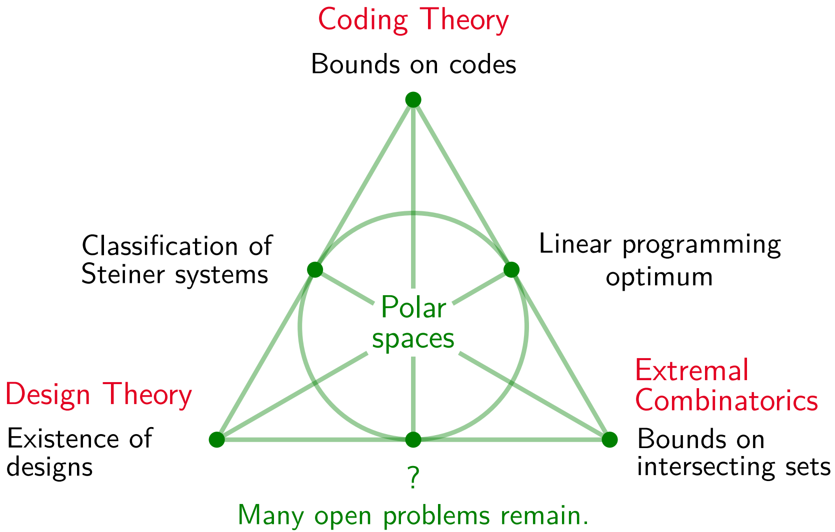

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Many open problems remain.