Near factorization of finite groups

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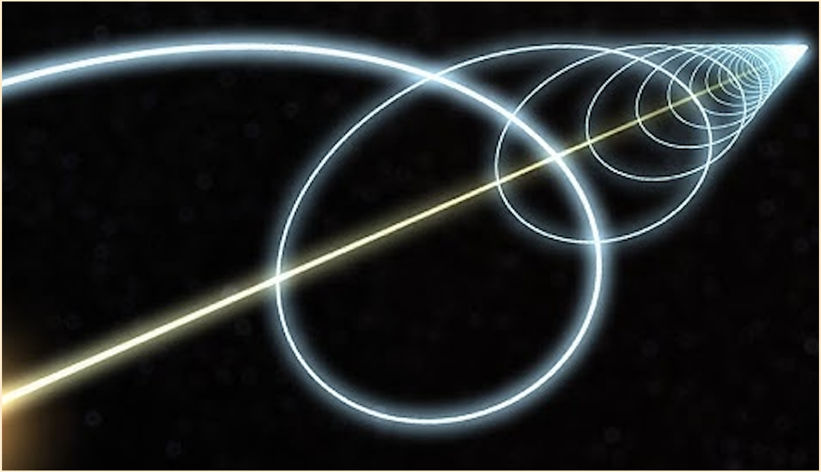


Definition

- Let $(G, \cdot)$ be a finite multiplicative group with identity e.
- For $S, T \subseteq G$, define $ST = \{gh : g \in A, h \in B\}$.
- We say that (S, T) is a near-factorization of G if

$$|S| \times |T| = |G| - 1 \quad \text{and} \quad G \setminus \{e\} = ST.$$

- In the case where we have an additive group $(G, +)$ with identity 0, the second condition becomes $G - 0 = S + T$.
- Further, (S, T) is a (s, t)-near-factorization of G if $|S| = s$ and $|T| = t$, which requires $st = |G| - 1$.
- There is always a trivial $(1, |G| - 1)$-near-factorization of G given by

$$S = \{e\}, \qquad T = G - e.$$

- A near-factorization with $|S| > 1$ and $|T| > 1$ is nontrivial.

Example 1: $\mathbb{Z}_{16}$

A $(3,5)$-near-factorization of $(\mathbb{Z}_{16},+)$ is given by

$$S = \{1,2,3\} \quad \text{and} \quad T = \{0,3,6,9,12\}.$$

We have

$$\begin{aligned} 1 + T &= \{1,4,7,10,13\} \\ 2 + T &= \{2,5,8,11,14\} \\ 3 + T &= \{3,6,9,12,15\} \end{aligned}$$

The union of these three sets is $\mathbb{Z}_{16} \setminus \{0\}$.

Example 2: $\mathbb{Z}_{1+rs}$

An (s,t)-near-factorization of $(\mathbb{Z}_{1+st}, +)$ is given by

$$S = \{1, 2, \ldots, s\} \quad \text{and} \quad T = \{0, s, 2s, \ldots, (t-1)s\}.$$

We have

$$\begin{aligned}
1 + T &= \{1, 1+s, 1+2s, \ldots, 1+(t-1)s\} \\
2 + T &= \{2, 2+s, 2+2s, \ldots, 2+(t-1)s\} \\
3 + T &= \{3, 3+s, 3+2s, \ldots, 3+(t-1)s\} \\
&\vdots \\
(s-1) + T &= \{(s-1), (s-1)+s, (s-1)+2s, \ldots, (s-1)+(t-1)s\} \\
s + T &= \{s, 2s, 3s, \ldots, st\}
\end{aligned}$$

The union of these s sets is $\mathbb{Z}_{1+st} \setminus \{0\}$.

Example 3 D_8

The dihedral group D_n of order $2n$, $n > 2$ has the presentation

$$D_n = \left\langle a, b : a^2 = b^n = abab = e \right\rangle,$$

where e is the identity element.

$$S = \{e, b, a\} \quad \text{and} \quad T = \{b^2, b^5, ab, ab^4, ab^7\}$$

form a $(3,5)$-near-factorization of the dihedral group D_8. We have

$$\begin{aligned} eT &= \{b^2, b^5, ab, ab^4, ab^7\} \\ bT &= \{b^3, b^6, a, ab^3, ab^6\} \\ aT &= \{ab^2, ab^5, b, b^4, b^7\}. \end{aligned}$$

The union of these three sets is $D_8 \setminus \{e\}$.

Example 4. D_n

We illustrate a general construction with $n = 13$.

- D_{13} can be depicted by the following diagram:

$i =$	0	1	2	3	4	5	6	7	8	9	10	11	12
b^i													
ab^i													

- Remove the identity and enter the sequence $1, 2, 3, 4, 5$ five times, as shown.

$i =$	0	1	2	3	4	5	6	7	8	9	10	11	12
b^i		1	2	3	4	5	1	2	3	4	5	1	2
ab^i	5	4	3	2	1	5	4	3	2	1	5	4	3

Example 4. continued

- Partition the cells into tiles of the same shape that each contain exactly one cell of each type.

$i =$	0	1	2	3	4	5	6	7	8	9	10	11	12
b^i		1	2	3	4	5	1	2	3	4	5	1	2
ab^i	5	4	3	2	1	5	4	3	2	1	5	4	3

- Let S be the group elements in the leftmost tile:

$$S = \{b, b^2, ab^2, ab, a\}.$$

Each tile has a "notch." Let T be the group elements corresponding to these notches:

$$T = \{e, ab^5, b^5, ab^{10}, b^{10}\}.$$

- Then $ST = D_{13} \setminus \{e\}$ and hence it is a $(5, 5)$-near-factorization.
- The same method of construction will produce a near-factorization of D_n into factors S and T, whenever $|S| \times |T| = 2n - 1$.

(0,1)-factorization of $J - I$

Example (S, T) a (2,2)-near-factorization of $\mathbb{Z}_5$,

Let $G = C_5$ with generator g. Take $S = \{g, g^2\}$ and $T = \{e, g^2\}$. Then

$$ST = \{g, g^2, g^3, g^4\} = C_5 - e.$$

Set $M_S[x, y] = \begin{cases} 1 & \text{if } x^{-1}y \in S \\ 0 & \text{otherwise;} \end{cases}$ $\quad M_T[x, y] = \begin{cases} 1 & \text{if } x^{-1}y \in T \\ 0 & \text{otherwise;} \end{cases}$

$$\underset{M_S}{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}} \underset{M_T}{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}} = \underset{J_5 - I_5}{\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}}$$

If (S, T) is a (k, ℓ) near-factorization of G, then $M_S M_T = J - I$

Partitionable graphs

- A graph H on $n = uv + 1$ vertices is (u, v)-partitionable if for every vertex x
  1. $H - x$ has a partition into u cliques of size v, and
  2. $H - x$ has a partition into v independent sets of size u.
- The construction uses Cayley graphs. Suppose G is a multiplicative group with identity e,
  - $S \subseteq G \setminus \{e\}$ is symmetric if $g^{-1} \in S$ whenever $g \in S$.
  - The Cayley graph with connection set S, denoted $\mathrm{Cay}(G, S)$, has vertex set G, and $\{x, y\}$ is an edge iff $x^{-1}y \in S$.

Note:

- Because S is symmetric, $x^{-1}y \in S$ iff $y^{-1}x \in S$.
- Because $e \notin S$, $x^{-1}x \notin S$, i.e. there are no loops.
- Hence because S is symmetric, $\mathrm{Cay}(G, S)$ is a graph (rather than a digraph).

Partitionable graphs Pêcher (2003)

- Suppose (S, T) is a near-factorization of G.
- Let $A = S^{-1}S \setminus \{e\} = \{x^{-1}y : x, y \in S, x \neq y\}$
- Then $\textsc{Cay}(G, A)$ has the following properties:
  1. $\textsc{Cay}(G, A)$ is vertex transitive:
     For each $g \in G$, $x \mapsto xg$ is an automorphism.
  2. $\textsc{Cay}(G, A)$ is normalized:
     for every edge xy, there is a max. clique containing $\{x, y\}$.
  3. $\textsc{Cay}(G, A)$ is partitionable:
     for every vertex $g \in G$, the induced subgraph that is obtained by deleting g, i.e., $\textsc{Cay}(G, A)[G \setminus \{g\}]$, has the partition

$$\{gbS : b \in T\} \text{ of } |T| \text{ cliques of size } |S|$$

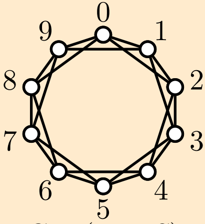
$$\{g(Ta)^{-1} : a \in S\} \text{ of } |S| \text{ independent sets of size } |T|$$

Example

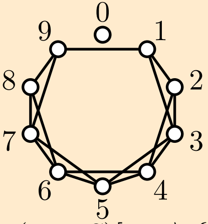
- Consider the near-factorization of $\mathbb{Z}_{10}$ given by $S = \{0, 1, 9\}$ and $T = \{2, 5, 8\}$. We have
$$-S + S = \{0, 1, 2, 8, 9\},$$
so $A = \{1, 2, 8, 9\}$.
- $\textsc{Cay}(\mathbb{Z}_{10}, A)$ is a graph whose vertices are $\mathbb{Z}_{10}$. So pairs of vertices that are distance 1 or 2 from each other are joined by edges.
- It is easy to see that $\textsc{Cay}(\mathbb{Z}_{10}, A)[\mathbb{Z}_{10} \setminus \{0\}]$ can be partitioned into three cliques of size three, namely
$$2 + S = \{1, 2, 3\},\ 5 + S = \{4, 5, 6\} \text{ and } 8 + S = \{7, 8, 9\}.$$
- It is also possible to partition $\textsc{Cay}(\mathbb{Z}_{10}, A)[\mathbb{Z}_{10} \setminus \{0\}]$ into three independent sets of size three, namely,
$$-(T + 0) = \{2, 5, 8\},\ -(T + 1) = \{1, 4, 7\},\ -(T + 9) = \{3, 6, 9\}.$$

Example

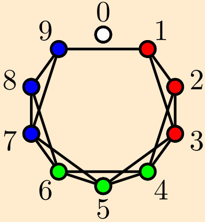
- Let $S = \{0, 1, 9\}$ and $T = \{2, 5, 8\}$. Then $S + T = \mathbb{Z}_{10} \setminus \{0\}$
- $S = (-S + S) \setminus \{0\} = \{1, 2, 8, 9\}$.



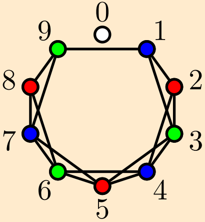
$\textsc{Cay}(\mathbb{Z}_{10}, S)$



$\textsc{Cay}(\mathbb{Z}_{10}, S)[\mathbb{Z}_{10} \setminus \{0\}]$



three cliques of size 3



three independent sets of size 3

Equivalence

Suppose (S,T) is a near-factorization of G. If $\alpha \in \textsc{Aut}(G)$ and $g \in G$, then

$$\begin{aligned}\big(\alpha(S)g\big)\big(g^{-1}\alpha(T)\big) &= \alpha(S)gg^{-1}\alpha(T) = \alpha(S)\alpha(T) = \alpha(ST)\\ &= \alpha(G \setminus \{e\}) = \alpha(G) \setminus \{\alpha(e)\}\\ &= G \setminus \{e\}\end{aligned}$$

Thus $(\alpha(S)g, g^{-1}\alpha(T))$ is an equivalent near-factorization of G.

A near-factorization (S,T) of an additive group G is symmetric if S and T are both symmetric.

Theorem 1 (de Caen et al, 1993).
If (S,T) is a near-factorization of additive abelian group G, then there exists $g \in G$ such that $(S+g, -g+T)$ is a symmetric near-factorization of G.

Every near-factorization of an abelian group is equivalent to a symmetric near-factorization.

Example

The $(3, 5)$-near-factorization of $\mathbb{Z}_{16}$ given by

$$S = \{0, 1, 15\} \quad \text{and} \quad T = \{2, 5, 8, 11, 14\}$$

is equivalent to the near-factorization

$$S' = 7S + 2 = \{2, 9, 11\} \quad \text{and} \quad T' = -2 + 7T = \{0, 1, 6, 11, 12\}$$

Theorem 1 guarantees that there is an element $g \in \mathbb{Z}_{16}$ such that $(S' + g, -g + T')$ is symmetric. The value $g = 14$ works, yielding

$$S' + 14 = \{0, 7, 9\} \quad \text{and} \quad -14 + T' = \{2, 3, 8, 13, 14\}.$$

Mates

- If S is a subset of the order n finite group G and T is such that (S, T) is a (r, s)-near-factorization of G, then we say T is a mate to S.
- If T is a mate to S, then $ST = G \setminus \{e\}$.
- Then $M_S M_T = J - I$, where $M_S[x, y] = \begin{cases} 1 & \text{if } x^{-1}y \in S; \\ 0 & \text{if not.} \end{cases}$
- Consequently $\det(J - I) = (-1)^{n-1}(n-1) \neq 0$.
- Thus $\det(M_S) \neq 0$
- Therefore

$$M_T = (M_S)^{-1}(J - I) = \frac{1}{r}J - (M_S)^{-1}$$

Theorem 2 (Kreher-Martin-Stinson 2025).
If $S \subseteq G$ has a mate T, then T is unique.

Computation

- Consider M_T

  $$M_T[x,y]=1 \Leftrightarrow x^{-1}y \in T \Leftrightarrow (yx^{-1})^{-1}e \in T \Leftrightarrow M_T[(yx^{-1}),e]=1$$

  The matrix M_T is completely determined by its "first" column.

- To determine if $S \subseteq G$ has a mate T we solve

  $$M_S X = [0,1,1,...,1]^T \quad (\text{ The first column of } J-I)$$

- If X exists and is a (0,1)-valued vector, then S has the mate T, where

  $$T=\{b^{-1} : X[b]=1\}$$

  (X is the first column of M_T.)

This is very efficient. However the number of possible subsets S to examine can be large.

reducing the search space

- The search space is the set of s element subsets $S \subseteq G \setminus \{e\}$ for which we compute a possible mate.
- If G is abelian we can assume the possible near-factorization are symmetric and only consider S, where $S = -S$.
- If we know $\textsc{Aut}(G)$ we need only consider S that are lexicographically minimal with respect to equivalence.

Computational results

- Near-factorizations of cyclic groups exist for all possible parameters. If $(n-1) = st$, then

$$\mathbb{Z}_n \setminus \{0\} = \{1, 2, \ldots, s\} + \{0, s, 2s, \ldots, (t-1)s\}$$

  See [3] for recent further results on this topic.

- For noncyclic abelian groups, it was previously known (mainly due to theoretical results in de Caen et al [1]) that there are no non-trivial examples in noncyclic abelian groups of order ≤ 100.

- We have now proven nonexistence in all noncyclic abelian groups G of order ≤ 200; there were roughly 100 parameter sets (G, r, s) to consider.
  - Most possibilities were ruled out by theoretical criteria, but several parameter sets required exhaustive searches.
  - "Difficult groups" requiring computer search: $\mathbb{Z}_{29} \times (\mathbb{Z}_2)^2$, $\mathbb{Z}_{17} \times (\mathbb{Z}_2)^3$, $\mathbb{Z}_{17} \times \mathbb{Z}_4\mathbb{Z}_2$, $\mathbb{Z}_{37} \times (\mathbb{Z}_2)^2$, $\mathbb{Z}_{17} \times (\mathbb{Z}_3)^2$, $\mathbb{Z}_{39} \times (\mathbb{Z}_2)^2$, $\mathbb{Z}_{19} \times (\mathbb{Z}_3)^2$, $\mathbb{Z}_{43} \times (\mathbb{Z}_2)^2$, $\mathbb{Z}_7 \times (\mathbb{Z}_5)^2$, $\mathbb{Z}_{11} \times \mathbb{Z}_8 \times \mathbb{Z}_2$, $\mathbb{Z}_{49} \times (\mathbb{Z}_2)^2$, and $\mathbb{Z}_{49} \times (\mathbb{Z}_2)^2$.

Nonabelian groups

The only known non-abelian groups that are known to have a near-factorization are:

- de Caen et al. The (s, t)-near-factorizations of the dihedral group D_n mentioned earlier,

  $$D_n = \langle a, b : a^2 = 1, b^n = 1, aba = b^{-1} \rangle$$

  for all $st = (n-1)$.
- Pêcher's $(7, 7)$-near-factorization of $D_5 \times C_5$.

  $$D_5 \times C_5 = \left\langle a, b, c : a^2 = b^5 = abab = c^5 = e, ac = ca, bc = cb \right\rangle .$$

- Pêcher's $(7, 7)$-near-factorization of $C_5^2 \rtimes_2 C_2$

  $$C_5^2 \rtimes_2 C_2 = \langle a, b, c | a^5 = b^5 = c^2 = e, cac = a^{-1}, cbc = b^{-1}, bc = cb \rangle$$

Pêcher checked all non-abelain groups of order at most 50.
See Kreher, Paterson and Stinson [4] and Pêcher [7].

λ-mates

- Let G be a finite group with identity e
- We say that (S, T) is a λ-fold near-factorization of G if $|S| \times |T| = \lambda(|G| \setminus \{e\})$ and each element of $G \setminus \{e\}$ occurs λ times in the **product** ST.

$$ST = \lambda(G \setminus \{e\})$$

- In the case where we have an additive group $(G, +)$ with identity 0, then each element of $G \setminus \{0\}$ occurs λ times in the **sum** $S + T$.

$$S + T = \lambda(G \setminus \{0\}).$$

- Necessarily $\lambda \leq s$ and $\lambda \leq t$.
- If (S, T) is a λ-fold near-factorization, then we say that T is a λ-mate of S.

What? They need not be symmetric?

There is an 2-fold $(3,4)$-near-factorization (S,T) of $\mathbb{Z}_7$.

$$S = \{0,1,3\} \quad \text{and} \quad T = \{1,2,3,5\}$$

$+$	1	2	3	5
0	1	2	3	5
1	2	3	4	6
3	4	5	6	1

There is not a symmetric 2-fold near-factorization (S,T) of $\mathbb{Z}_7$.

Proof.

Let (S,T) be a symmetric 2-fold $(3,4)$-near factorization of $\mathbb{Z}_7$.

- $|S| = 3 \Rightarrow S = \{0,x,-x\}$.
- (S',T'), where $S' = Sx^{-1} = \{0,1,-6\}$ and $T' = xT$ is also a 2-fold near-factorization of $\mathbb{Z}_7$.
- Because $0 \notin S' + T' \Rightarrow 0,1,6 \notin T' \Rightarrow T' = \{2,3,4,5\}$
- But $S' + T'$ contains $0+3 = 1+2 = 6+4 = 3$, and 3 should occur twice. □

λ-fold(s,t) near factorizations with $\lambda > 2$, $n \leq 35$

Symmetric

n	group	s	t	λ
9	$(\mathbb{Z}_3)^2$	4	4	2
13	$\mathbb{Z}_{13}$	6	6	3
15	$\mathbb{Z}_{15}$	4	7	2
16	$(\mathbb{Z}_4)^2$	6	10	4
16	$(\mathbb{Z}_2)^4$	6	10	4
17	$\mathbb{Z}_{17}$	8	8	4
21	$\mathbb{Z}_{21}$	4	10	2
25	$(\mathbb{Z}_5)^2$	4	12	2
25	$(\mathbb{Z}_5)^2$	12	12	6
27	$\mathbb{Z}_9 \times \mathbb{Z}_3$	4	13	2
27	$(\mathbb{Z}_3)^3$	8	13	4
29	$\mathbb{Z}_{29}$	14	14	7
33	$\mathbb{Z}_{33}$	4	16	2
35	$\mathbb{Z}_{35}$	4	17	2

Non-symmetric

n	group	s	t	λ
7	$\mathbb{Z}_7$	3	4	2
11	$\mathbb{Z}_{11}$	5	6	3
13	$\mathbb{Z}_{13}$	4	9	3
15	$\mathbb{Z}_{15}$	7	8	4
16	$\mathbb{Z}_8 \times \mathbb{Z}_2$	5	9	3
16	$\mathbb{Z}_8 \times \mathbb{Z}_2$	6	10	4
16	$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	6	10	4
19	$\mathbb{Z}_{19}$	9	10	5
21	$\mathbb{Z}_{21}$	5	16	4
21	$\mathbb{Z}_{21}$	8	10	4
23	$\mathbb{Z}_{23}$	11	12	6
27	$(\mathbb{Z}_3)^3$	13	14	7
28	$\mathbb{Z}_{14} \times \mathbb{Z}_2$	9	12	4
31	$\mathbb{Z}_{31}$	6	20	4
31	$\mathbb{Z}_{31}$	6	25	5
35	$\mathbb{Z}_{35}$	8	17	4
35	$\mathbb{Z}_{35}$	17	18	9

If $S \subseteq G$, then $S^{-1} = \{x^{-1} : x \in S\}$.
(If G is abelian and written additively, $S^{-1} = -S = \{-x : x \in S\}$.)

A (v, k, λ)-difference set in the group G is a k-element subset S of G such that the identity e occurs k times in the product SS^{-1} and each non-identity element occurs λ times.

Theorem 3. Suppose there is a (v, k, λ)-difference set S in a group G of order v. If $T = G \setminus S^{-1}$. Then (S, T) is a $(k - \lambda)$-fold $(k, v - k)$-near factorization.

Example
A $(11, 5, 2)$-difference set in $\mathbb{Z}_{11}$ is
$S = \{1, 3, 4, 5, 9\}$

$S^{-1} = -S = \{10, 8, 7, 6, 2\} \Rightarrow$
$T = \{0, 1, 3, 4, 5, 9\}$

$S + T = 3(\mathbb{Z}_{11} \setminus \{0\})$

$-$	1	3	4	5	9
1	0	9	8	7	3
3	2	0	10	9	6
4	3	1	0	10	5
5	4	2	1	0	7
9	8	6	5	4	0

$+$	0	1	3	4	5	9
1	1	2	4	5	6	10
3	3	4	6	7	8	1
4	4	5	7	8	9	2
5	5	6	8	9	10	3
9	9	10	1	2	3	7

The group ring $\mathbb{Z}[G]$.

Let G be a finite group. The group ring $\mathbb{Z}[G]$ is

$$\mathbb{Z}[G] = \{ \sum_{g \in G} c_g g : c_g \in \mathbb{Z}, g \in G \}$$

Then the multi-subset S of G, is denoted in the group ring as $S = \sum_{g \in S} n_j g$, where n_g is the number of times g occurs in S

Example: $G = C_7 = \{1, \alpha, \alpha^2, \cdots, \alpha^6\}$, the cyclic group of order 7 generated by α. Then

$$\{1, \alpha, \alpha, \alpha^3\} \text{ in } G \equiv 1 + 2\alpha + \alpha^3 \in \mathbb{Z}[G]$$

addition:

$$(1 + \alpha + \alpha^5) + (\alpha + \alpha^6) = (1 + 2\alpha + \alpha^5 + \alpha^6)$$

mutiplication:

$$\begin{aligned} (1 + \alpha + \alpha^5)(\alpha + \alpha^6) &= 1(\alpha + \alpha^6) + \alpha(\alpha + \alpha^6) + \alpha^5(\alpha + \alpha^6) \\ &= (\alpha + \alpha^6) + (\alpha^2 + e) + (\alpha^6 + \alpha^4) \\ &= 1 + \alpha + \alpha^2 + \alpha^4 + 2\alpha^6 \end{aligned}$$

The group ring $\mathbb{Z}[G]$. Continued

If $S \subset G$, let $S = \sum_{g \in S} g$, then (S, T) is a λ-fold near-factorization of G, if and only if in the group ring

$$ST = \lambda(G - e)$$

Example in $\mathbb{Z}[C_7]$:

$$\begin{aligned}
(e + \alpha + \alpha^3)(\alpha + \alpha^2 + \alpha^3 + \alpha^5) &= e(\alpha + \alpha^2 + \alpha^3 + \alpha^5) \\
&\quad +\alpha(\alpha + \alpha^2 + \alpha^3 + \alpha^5) \\
&\quad +\alpha^3(\alpha + \alpha^2 + \alpha^3 + \alpha^5) \\
&= (\alpha + \alpha^2 + \alpha^3 + \alpha^5) \\
&\quad +(\alpha^2 + \alpha^3 + \alpha^4 + \alpha^6) \\
&\quad +(\alpha^4 + \alpha^5 + \alpha^6 + \alpha) \\
&= 2(G_7 - 1)
\end{aligned}$$

The group ring $\mathbb{Z}[G]$ is a convenient algebraic way to handle multi-sets.

Proof of Theorem 3

If $S \subset G$, then $S^{(-1)} = \sum_{g \in S} g^{-1}$.

A k-element subset $D \subseteq G$ is a (v, k, λ)-difference set if and only if

$$DD^{(-1)} = ke + \lambda(G - e)$$

Theorem 3. Suppose there is a (v, k, λ)-difference set D in a group G of order v. If $S = D$ and $T = G \setminus S^{-1} = \{g \in G : g^{-1} \notin S\}$, then (S, T) is a $(k - \lambda)$-fold $(k, v - k)$-near factorization.

Proof.

First: $SG = kG$

Next: $SS^{(-1)} = ke + \lambda(G - e)$

Hence:
$$\begin{aligned} ST &= S(G - S^{(-1)}) \\ &= kG - \big(ke + \lambda(G - e)\big) \\ &= (k - \lambda)(G - e) \end{aligned}$$

□

Remark

Suppose the k-element subset $S \subseteq G$ is a (v, k, λ)-difference set then

$$SS^{(-1)} = ke + \lambda(G - e)$$

- **The "inverse" is also a difference set.**

  $$S^{(-1)}S = ke + \lambda(G - e)$$

  So $S^{(-1)}$ is also a difference set.

- **The complement is also a difference set**
  Let $T = G \setminus S$, where S is a a (v, k, λ)-difference set, let $t = |T| = v - k$.

  $$\begin{aligned} TT^{(-1)} &= (G - S^{(-1)})(G - S^{(-1)})^{(-1)} = (G - S^{(-1)})(G - S) \\ &= GG - GS - S^{(-1)}G + S^{(-1)}S \\ &= (v)G - kG - kG + \big(ke + \lambda(G - e)\big) \\ &= (t - k)G + ke + \lambda(G - e) \\ &= (t + \lambda - k)(G - e) + te \end{aligned}$$

The converse is true

Theorem 3 converse.
Suppose (S, T) is λ-fold (s, t)-near factorization of G, where $|G| = s + t$. Then S is an $(s + t, s, s - \lambda)$-difference set in G and $T = G \setminus S^{-1}$ is an $(s + t, t, t - \lambda)$-difference set in G.

Proof.

In the $\mathbb{Z}[G]$, $T = G - S^{(-1)}$.

$$SS^{(-1)} = S(G - T) = sG - ST = sG - \lambda(G - e)$$
$$= (sG - e) + se - \lambda(G - e) = (s - \lambda)(G - e) + se$$

Therefore S is a $(s + t, s, s - \lambda)$-difference set.
and T is a $(s + t, t, t - \lambda)$-difference set,
because T is the complement of S^{-1}. □

Partial difference set

A (v, k, λ, μ)-partial difference set (or PDS) in a group G of order v is a subset $D \subseteq G \setminus \{e\}$ such that $|D| = k$ and the following group ring equation is satisfied:

$$\begin{aligned} DD^{(-1)} &= (k-\mu)e + (\lambda-\mu)D + \mu G, \\ &= ke + \lambda D + \mu(G - D - e) \end{aligned}$$

The set $D = \{1, 3, 4, 9, 10, 12\}$ is a $(13, 6, 2, 3)$-PDS in $\mathbb{Z}_{13}$.

$-$	1	3	4	9	10	12
1	0	11	10	5	4	2
3	2	0	12	7	6	4
4	3	1	0	8	7	5
9	8	6	5	0	12	10
10	9	7	6	1	0	11
12	11	9	8	3	2	0

PDS construction

Theorem 4. Suppose D is a $(s+t+1, s, s-\lambda-1, s-\lambda)$-PDS in a group G, where $|G| = s+t+1$ and $e \notin D$. Let $S = D$ and $T = G \setminus S^{(-1)} \setminus \{e\}$. Then (S, T) is a λ-fold (s, t)-near-factorization of G.

Proof.

Computing in $\mathbb{Z}[G]$ we see that

$$\begin{aligned} ST &= S\big(G - S^{(-1)} - e\big) \\ &= SG - SS^{(-1)} - Se \\ &= sG - \Bigg(se + (s-\lambda-1)S + (s-\lambda)(G - S - e)\Bigg) - S \\ &= sG - se - (s-\lambda-1)S - (s-\lambda)(G - S - e) - S \\ &= \lambda G - \lambda e \\ &= \lambda(G - e) \end{aligned}$$

□

Example and converse

From the (13,6,2,3)-PDS given in the Example a 3-fold (6,6)-near-factorization of $\mathbb{Z}_{13}$ is obtained. The near-factorization has

$$S = \{1, 3, 4, 9, 10, 12\} \text{ and } T = \{2, 5, 6, 7, 8, 11\}.$$

Theorem 4 converse.
If (S, T) is an λ-fold (s, t)-near-factorization of G and $|G| = s + t + 1$. Then S is an $(s + t + 1, s, s - \lambda - 1, s - \lambda)$-PDS in G and T is an $(s + t + 1, t, t - \lambda - 1, t - \lambda)$-PDS

Theorem 5. Suppose p and q are any positive odd integers greater than 1. Then there exists a 2-fold $(4,(n-1)/2)$-near-factorization (S,T) of $\mathbb{Z}_p \times \mathbb{Z}_q$.

The construction: Take

$$S = \{(1,1),(1,-1),(-1,1),(-1,-1)\}.$$

Set $C_i^j = \{4i+j, 4i+j+1\}$.

Case 1: $p = 1+4a$, $q = 1+4b$.

$$T = \left(\left(\bigcup_{i=0}^{a-1} C_i^0 \cup \{4a\}\right) \times \left(\bigcup_{j=0}^{b-1} C_j^2\right)\right) \cup \left(\left(\bigcup_{i=0}^{a-1} C_i^2\right) \times \left(\bigcup_{j=0}^{b-1} C_j^0 \cup \{4b\}\right)\right)$$

Case 2: $p = 1+4a$, $q = -1+4b$.

$$T = \left(\left(\bigcup_{i=0}^{a-1} C_i^0 \cup \{4a\}\right) \times \left(\bigcup_{j=0}^{b-2} C_j^3 \cup \{0\}\right)\right) \cup \left(\left(\bigcup_{i=0}^{a-1} C_i^2\right) \times \left(\bigcup_{j=0}^{b-1} C_j^1\right)\right)$$

Case 3: $p = -1+4a$, $q = -1+4b$.

$$T = \left(\left(\bigcup_{i=0}^{a-2} C_i^3 \cup \{0\}\right) \times \left(\bigcup_{j=0}^{b-1} C_j^1\right)\right) \cup \left(\left(\bigcup_{i=0}^{a-1} C_i^1\right) \times \left(\bigcup_{j=0}^{b-2} C_j^3 \cup \{0\}\right)\right)$$

Example $\mathbb{Z}_{45} = \mathbb{Z}_5 \times \mathbb{Z}_9$

$S = \{(1,1), (1,8), (4,1), (4,8)\}.$

$$C_0^0 = \{0,1\} \qquad C_1^0 = \{4,5\}.$$
$$C_0^2 = \{2,3\} \qquad C_1^2 = \{6,7\}.$$

Case 1: $p = 1 + 4a$, $q = 1 + 4b$, where $a = 1$, $b = 2$

$$\begin{aligned}
T &= \left(\left(\bigcup_{i=0}^{a-1} C_i^0 \cup \{4a\}\right) \times \left(\bigcup_{j=0}^{b-1} C_j^2\right)\right) \cup \left(\left(\bigcup_{i=0}^{a-1} C_i^2\right) \times \left(\bigcup_{j=0}^{b-1} C_j^0 \cup \{4b\}\right)\right) \\
&= \left(\left(C_0^0 \cup \{4\}\right) \times \left(C_0^2 \cup C_1^2\right)\right) \cup \left(\left(C_0^2\right) \times \left(C_0^0 \cup C_1^0 \cup \{8\}\right)\right) \\
&= \left(\{0,1,4\} \times \{2,3,6,7\}\right) \cup \left(\{2,3\} \times \{0,1,4,5,8\}\right) \\
&= \left\{ \begin{matrix} (0,2), (0,3), (0,6), (0,7), (1,2), (1,3), (1,6), (1,7), (4,2), (4,3), (4,6) \\ (4,7), (2,0), (2,1), (2,4), (2,5), (2,8), (3,0), (3,1), (3,4), (3,5), (3,8) \end{matrix} \right\}
\end{aligned}$$

Example Continued

Thus

$$S = \{(1,1),(1,8),(4,1),(4,8)\}.$$
$$T = \begin{Bmatrix} (0,2),(0,3),(0,6),(0,7),(1,2),(1,3),(1,6),(1,7),(4,2),(4,3),(4,6) \\ (4,7),(2,0),(2,1),(2,4),(2,5),(2,8),(3,0),(3,1),(3,4),(3,5),(3,8) \end{Bmatrix}$$

is (supposedly) a 2-fold $(4,22)$-Near-factorization of $\mathbb{Z}_5 \times \mathbb{Z}_9$.
$(1,1)$ generates $\mathbb{Z}_5 \times \mathbb{Z}_9$ and 1 generates $\mathbb{Z}_{45}$ and so $\psi : (x,x) \mapsto x$ is an isomorphism. For example $(1,8) = (26,26)$ So $\pi(1,8) = 26$.
Thus

$$\psi(S) = \{1, 26, 19, 44\}.$$
$$\psi(T) = \begin{Bmatrix} 6, 8, 11, 13, 15, 16, 17, 18, 20, 21, 22, \\ 23, 24, 25, 27, 28, 29, 30, 32, 34, 37, 39 \end{Bmatrix}$$

is (supposedly) a 2-fold $(4,22)$-Near-factorization of $\mathbb{Z}_{45}$.

Lets check!

Example continued

$$S' = \psi(S) = \{1, 26, 19, 44\}.$$

$$T' = \psi(T) = \begin{Bmatrix} 6, 8, 11, 13, 15, 16, 17, 18, 20, 21, 22, \\ 23, 24, 25, 27, 28, 29, 30, 32, 34, 37, 39 \end{Bmatrix}$$

is (supposedly) a 2-fold $(4, 22)$-Near-factorization of $\mathbb{Z}_{45}$.

Lets check!

	6	8	11	13	15	16	17	18	20	21	22	23	24	25	27	28	29	30	32	34	37	39
1	7	9	12	14	16	17	18	19	21	22	23	24	25	26	28	29	30	31	33	35	38	40
19	25	27	30	32	34	35	36	37	39	40	41	42	43	44	1	2	3	4	6	8	11	13
26	32	34	37	39	41	42	43	44	1	2	3	4	5	6	8	9	10	11	13	15	18	20
44	5	7	10	12	14	15	16	17	19	20	21	22	23	24	26	27	28	29	31	33	36	38

and it is.

n	group	s	t	λ	Sym.?	Authority
7	$\mathbb{Z}_7$	3	4	2	no	Theorem 3, $D = \{$ 0, 1, 3$\}$
9	$(\mathbb{Z}_3)^2$	4	4	2	yes	Theorem 5
11	$\mathbb{Z}_{11}$	5	6	3	no	Theorem 3, $D = \{$ 0, 1, 2, 4, 7$\}$
13	$\mathbb{Z}_{13}$	4	9	3	no	Theorem 3, $D = \{$ 0, 1, 3, 9$\}$
13	$\mathbb{Z}_{13}$	6	6	3	yes	Theorem 4, $D = \{$1, 3, 4, 12, 10, 9$\}$
15	$\mathbb{Z}_{15}$	4	7	2	yes	Theorem 5
15	$\mathbb{Z}_{15}$	7	8	4	no	Theorem 3, $D = \{$ 0, 1, 2, 4, 5, 8, 10$\}$
16	$(\mathbb{Z}_4)^2$	6	10	4	yes	Theorem 3, $D = \{$(0,1), (1,0), (1,1), (0,3), (3,0), (3,3)$\}$
16	$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	6	10	4	yes	Theorem 3, $D = \{$(0,0,0), (0,0,1), (0,1,0), (2,1,1), (1,0,0), (3,0,0)$\}$
16	$\mathbb{Z}_8 \times \mathbb{Z}_2$	5	9	3	no	$S = \{$(0,0), (0,1), (1,0), (3,0), (4,0)$\}$, $T = \{$(7,1), (6,0), (5,1), (4,1), (3,0), (3,1), (2,0), (1,0), (1,1)$\}$
16	$\mathbb{Z}_8 \times \mathbb{Z}_2$	6	10	4	no	Theorem 3, $D = \{$(0,0), (0,1), (1,0), (2,0), (5,0), (6,1)$\}$
17	$\mathbb{Z}_{17}$	8	8	4	yes	Theorem 4, $D = \{$1, 2, 4, 8, 16, 15, 13, 9$\}$
19	$\mathbb{Z}_{19}$	9	10	5	no	Theorem 3, $D = \{$ 0, 1, 2, 3, 5, 7, 12, 13, 16$\}$
21	$\mathbb{Z}_{21}$	4	10	2	yes	Theorem 5
21	$\mathbb{Z}_{21}$	5	16	4	no	Theorem 3, $D = \{$ 0, 1, 4, 14, 16$\}$
21	$\mathbb{Z}_{21}$	8	10	4	no	$S = \{$ 0, 1, 3, 6, 7, 10, 13, 15$\}$, $T = \{$ 17, 13, 12, 9, 7, 5, 4, 3, 2, 1$\}$
23	$\mathbb{Z}_{23}$	11	12	6	no	Theorem 3, $D = \{$ 0, 1, 2, 3, 5, 7, 8, 11, 12, 15, 17$\}$

n	group	s	t	λ	Sym.?	Authority
25	$(\mathbb{Z}_5)^2$	4	12	2	yes	Theorem 5
25	$(\mathbb{Z}_5)^2$	12	12	6	yes	Theorem 4, $D=\{$(0,1), (0,2), (1,0), (1,1), (2,0), (2,2), (0,4), (0,3), (4,0), (4,4), (3,0), (3,3)$\}$
27	$(\mathbb{Z}_3)^3$	8	13	4	yes	$S = \{$(0,0,1), (0,1,0), (1,0,0), (1,1,1), (0,0,2), (0,2,0), (2,0,0), (2,2,2)$\}$, $T = \{$(0,0,0), (0,2,1), (0,1,2), (2,0,1), (2,2,1), (2,1,0), (2,1,2), (2,1,1), (1,0,2), (1,2,0), (1,2,2), (1,2,1), (1,1,2)$\}$
27	$(\mathbb{Z}_3)^3$	13	14	7	no	Theorem 3, $D=\{$(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,2,0), (1,0,0), (1,0,1), (1,1,0), (2,0,1), (2,1,2), (2,2,0), (2,2,2)$\}$
27	$\mathbb{Z}_9\times\mathbb{Z}_3$	4	13	2	yes	Theorem 5
28	$\mathbb{Z}_{14}\times\mathbb{Z}_2$	9	12	4	no	$S=\{$(0,0), (0,1), (1,0), (2,0), (3,1), (4,1), (7,1), (12,0), (13,0)$\}$, $T=\{$(13,1), (12,1), (11,0), (9,0), (9,1), (8,1), (6,1), (5,0), (4,1), (3,0), (3,1), (1,1)$\}$
29	$\mathbb{Z}_{29}$	14	14	7	yes	Theorem 4, $D=\{$1, 4, 5, 6, 7, 9, 13, 28, 25, 24, 23, 22, 20, 16$\}$
31	$\mathbb{Z}_{31}$	6	20	4	no	$S=\{$ 0, 1, 2, 4, 8, 16$\}$, $T=\{$ 28, 26, 25, 24, 22, 21, 19, 17, 16, 14, 13, 12, 11, 8, 7, 6, 4, 3, 2, 1$\}$
31	$\mathbb{Z}_{31}$	6	25	5	no	Theorem 3, $D=\{$ 0, 1, 3, 8, 12, 18$\}$
31	$\mathbb{Z}_{31}$	15	16	8	no	Theorem 3, $D=\{$ 0, 1, 2, 3, 5, 6, 8, 9, 13, 16, 21, 22, 23, 25, 27$\}$
33	$\mathbb{Z}_{33}$	4	16	2	yes	Theorem 5
33	$\mathbb{Z}_{33}$	12	16	6	no	$S=\{$ 0, 1, 3, 4, 6, 10, 12, 15, 21, 22, 25,

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Thank you