

On Boolean Degree 1 Functions, Anti-Designs, and Cameron-Liebler Sets in Finite Vector Spaces

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A Simple Question

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an **affine function**, that is

$$f(x) = c + \sum c_i x_i.$$

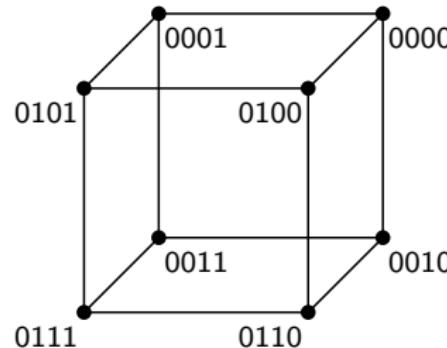
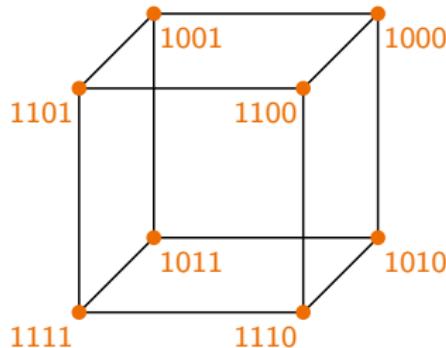
Definition

We call f **Boolean** over D if $f(x) \in \{0, 1\}$ for all $x \in D$.

Question

*What are the Boolean affine functions for the **hypercube** $D = \{0, 1\}^n$?*

Dictator



Polynomial f : $f(x) = x_1$.

We only need x_1 to determine $f(x)$.

The Degree 1 Functions on the Hypercube

Example

The constant functions $f(x) = 0$ and $f(x) = 1$.

Example

The functions $f(x) = x_i$ and $f(x) = 1 - x_i$.

Proposition

Let f be an **affine Boolean function** on the **hypercube**.

Then $f(x) = c + \sum c_i x_i$ is one of 0 , 1 , x_i , or $1 - x_i$.

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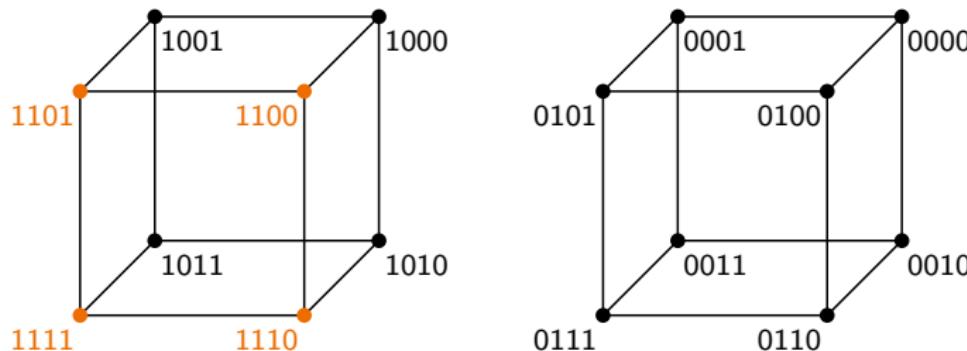
Proof.

WLOG $f(00\dots 0) = 0$. Hence, $c = 0$.

WLOG $f(10\dots 0) = 1$. Hence, $c_1 = 1$.

Now all the other c_i 's must be 0. □

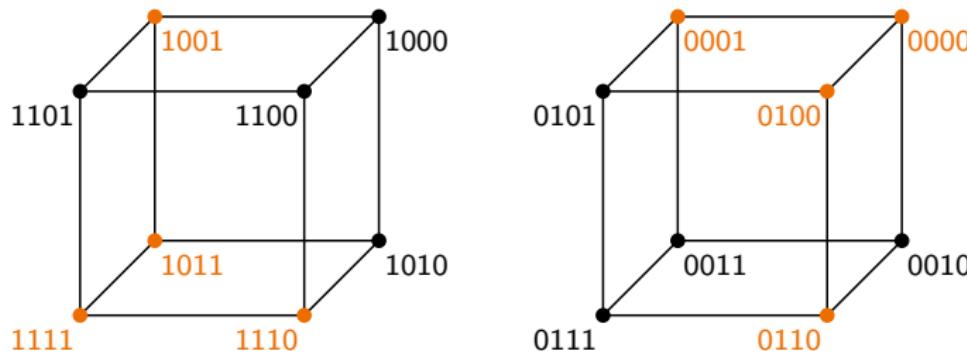
Degree 2, Example 1



Polynomial f : $f(x) = x_1x_2$.

We only need x_1, x_2 to determine $f(x)$.

Degree 2, Example 2



Polynomial f : $f(x) = 1 - x_1 + (x_1 + x_2 - 1)x_3 + (x_1 - x_2)x_4.$

We need x_1, x_2, x_3, x_4 to determine $f(x)!$

Some Generalizations (I)

Question

What about Boolean degree 2 functions f on the hypercube?

Answer: Then f or $1 - f$ is one of these:¹

- 0,
- x_i ,
- $x_i + x_j - x_i x_j$,
- $x_i x_j + (1 - x_i) x_k$,
- $x_i x_j + x_i x_k + x_j x_k - x_i - x_j - x_k$,
- $f(x) = 1$ iff $x_i \leq x_j \leq x_k \leq x_\ell$ or $x_i \geq x_j \geq x_k \geq x_\ell$.

Degree 2: Camion, Carlet, Charpin & Sendrier (1991).

Degree 3: Kirienko (2004), Zverev (2008).

¹I stole this list from Yuval Filmus.

Some Generalizations (II)

Question

What about Boolean degree d functions f on the hypercube?

Only m relevant variables: m -junta.

Theorem (Nisan, Szegedy (1991))

A Boolean degree d function on the hypercube is a $d \cdot 2^{d-1}$ -junta.

Chiarelli, Hatami and Saks (2018): Tight bound of $O(2^d)$.

Current best by Wellens (2019): $\leq 4.416 \cdot 2^d$.

Carlet and Tarannikov (2002): Lower bound of $3 \cdot 2^{d-1} - 2$.

Two applications: cryptography, complexity theory.

Some Generalizations (III)

Question

*What about **almost** affine Boolean functions f on the hypercube?*

That is, $\|f - g\|_2 < \varepsilon$ for some affine function g .

Friedgut, Kalai, and Naor (2002): If f is Boolean and almost affine, then f is almost a Boolean affine function.

Kindler, Safra (2002): Similar result for degree d .

Some Other Words

We already **know**: Boolean degree 1 function.

Other used words which might mean the same (depending on **context**):

- Equitable bipartition.
- Regular set.
- Perfect 2-coloring.
- **Cameron-Liebler set.**
- Completely regular code.
- Tight set.
- **Anti-1-design.**²
- Dual degree 1.
- Graphical design.
- Intriguing set.

²Boolean degree d function = design-orthogonal to a d -design (Delsarte theory)

“What should we do then?” Luke 3:10, NIV⁴

In the **hypercube**: Good understanding of low degree functions.

What about **other domains**?

For instance:

- A **slice** of the hypercube: all k -sets of $\{1, \dots, n\}$ (Johnson graphs).
- The **q -analog** of the slice: all k -spaces of \mathbb{F}_q^n (Grassmann graphs).

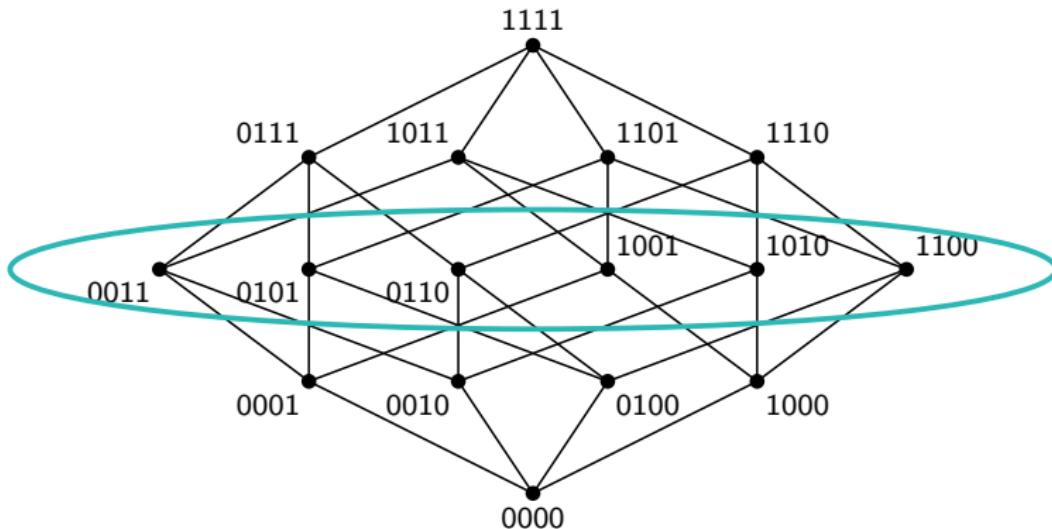
We will look at **k-sets** and **k-spaces**.³

³Cf. Kiermaier, Mannaert, Wassermann (2024).

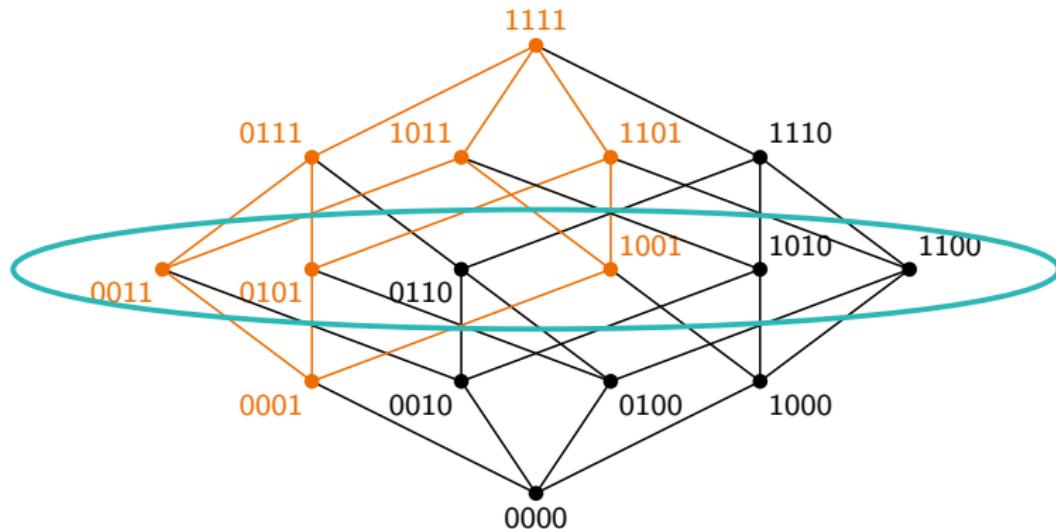
⁴Alternative quotes: “What is to be done?”.

Tolstoy, 1866; Lenin, 1902.

Subsets



Subsets



Theorem

Boolean degree 1 functions on k -sets of $\{1, \dots, n\}$ are **trivial**.
 I.e. they are **dictators** ($0, 1, x_i$ or $1 - x_i$). (For $n-k, k \geq 2$.)

Various proofs: Meyerowitz (1992, see Martin (2004)), Filmus (2016), Filmus and Ih. (2019). Also De Boeck, Strome, Svob (2016), but only for $k \mid n$.

Bounded Degree

Recall FKN for hypercube:

Boolean almost degree 1 \rightarrow almost dictator.

For k -sets of $\{1, \dots, n\}$:

Theorem (Filmus (2016))

Boolean almost degree 1 \rightarrow almost sum of dictators (or complement).

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Recall for hypercube: Boolean degree $d \rightarrow \gamma(d)$ -junta.

Theorem (Filmus, Ih. (2019))

If $\min(k, n - k) \geq C^d$: Boolean degree $d \rightarrow \gamma(d)$ -junta.

Keller, Klein (2019): stability. Filmus, Vinciguerra (unpub.): $n - k \geq n_0(d)$, $k \geq 2d$.

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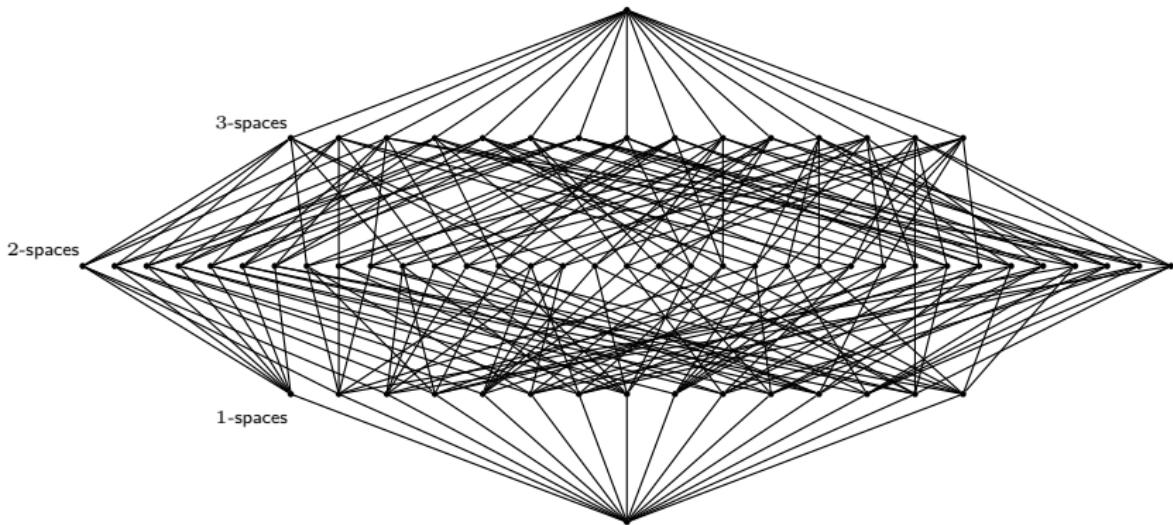
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Theorem (Filmus (2023))

If $\min(k, n - k) \geq 2d$: Boolean degree $d \rightarrow \gamma'(d)$ -junta.

Note: We have $\gamma(2) = 4$, but there is an example in $J(8, 4)$ with 5 relevant variables

Vector Spaces



The subspace lattice of \mathbb{F}_2^4 .

We consider ***k-spaces*** of a finite vector space!

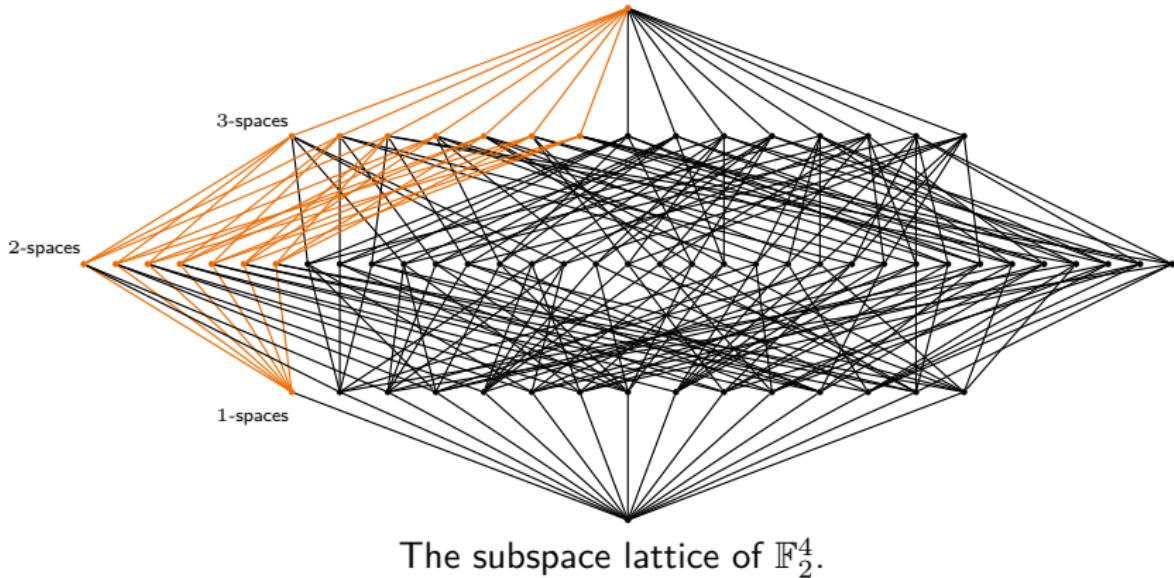
Degree 1: $f = \sum_P c_P x_P$, P 's are 1-spaces.

Here $x_P(S) = 1$ if $P \subseteq S$ and $x_P(S) = 0$ otherwise.

Degree 1, alternative: $f = \sum_H c_H x_H$, H 's are $(n - 1)$ -spaces.

Here $x_H(S) = 1$ if $S \subseteq H$ and $x_H(S) = 0$ otherwise.

Trivial Degree 1 in Vector Spaces (I)



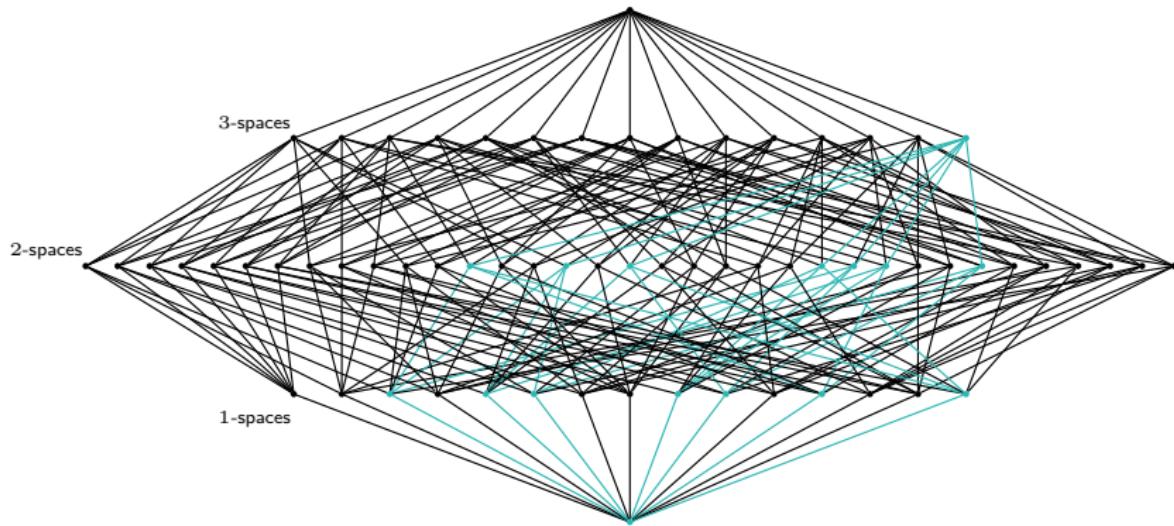
The subspace lattice of \mathbb{F}_2^4 .

Example (Trivial Example 1)

Take all k -spaces through a fixed **1-space** P : x_P .

Or the complement: $1 - x_P$. (This is always possible.)

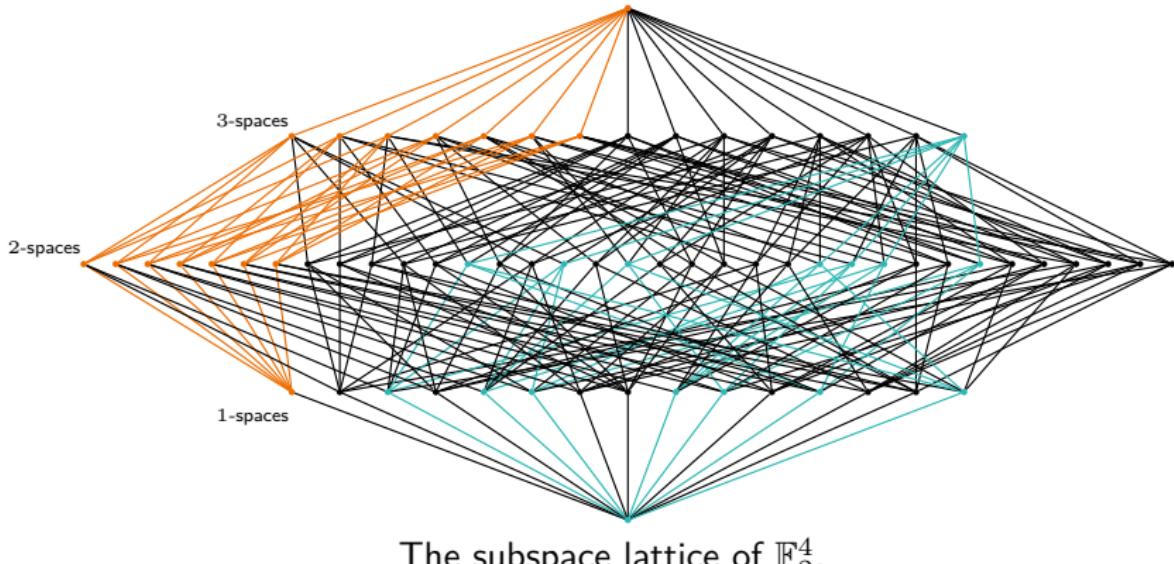
Trivial Degree 1 in Vector Spaces (II)

The subspace lattice of \mathbb{F}_2^4 .

Example (Trivial Example 2)

Take all k -spaces in a fixed **hyperplane** H : x_H .Degree 1 in x_P 's? Write $H = \alpha \sum_{P \subseteq H} x_P + \beta \sum_{P \not\subseteq H} x_P$.

Trivial Degree 1 in Vector Spaces (III)



Example (Trivial Example 3)

All through **1-space P or in hyperplane H** : $x_P + x_H$.

Or the complement: $1 - x_P - x_H$.

Degree 1 Functions on 2-spaces in \mathbb{F}_q^n

Cameron, Liebler (1982): Investigate action of subgroups of $PTL(4, q)$ on 1- and 2-spaces of \mathbb{F}_q^4 .

Same number of orbits: Boolean degree 1 function.

Degree 1 Functions on 2-spaces in \mathbb{F}_q^n

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Same number of orbits: Boolean degree 1 function.

Conjecture (Cameron, Liebler (1982, very simplified))

If Boolean degree 1 function f on 2-spaces, then f or $1 - f$ is ...

- 0,
- x_P for a 1-space P ,
- x_H for a hyperplane H , or
- $x_P + x_H$ for a 1-space P and a hyperplane H , $P \not\subseteq H$.

- **Conjecture very natural:** true for subsets.
- **True** for 2-spaces of \mathbb{F}_2^n by Drudge (1998).
- **False** for 2-spaces of \mathbb{F}_q^4 : First counterexample for $q = 3$ by **Drudge** (1998), later many more for $(n, k) = (4, 2)$.

State of the Art

For 2-spaces in \mathbb{F}_q^4 many **counterexamples**, e.g.:

Bruen, Cossidente, De Beule, Demeyer, Drudge, Feng, Gavrilyuk, Matkin, Metsch, Momihara, Pavese, Penttila, Rodgers, Xiang, Zou.

Restrictions on sizes of non-trivial examples for 2-spaces in \mathbb{F}_q^4 , e.g.:

- Metsch (2010),
- Metsch (2014),
- Gavrilyuk, Metsch (2014).

Restrictions in a **more general setting**:

- Metsch (2017),
- Rodgers, Storme, Vansweevelt (2018),
- Blokhuis, De Boeck, D'haeseleer (2019),
- De Beule, Mannaert, Storme (2022),
- Ihringer (2024),
- De Beule, Mannaert, Storme (2024).

Classification Results

Boolean degree 1 functions f on k -spaces for $n > 4$:

Theorem (Drudge (1998), Gavrilyuk and Mogilnykh (2014), Gavrilyuk and Matkin (2018), Matkin (2018))

All trivial for $k = 2$ and $q \in \{2, 3, 4, 5\}$.

Proof: Clever computations and induction on n .

Theorem (Filmus, Ih. (2019))

All trivial for $k \geq 2$ and $q \in \{2, 3, 4, 5\}$.

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Theorem (Ih. (2024, AMS Proceedings))

All trivial for $k \geq 2$ and $\max(n - k, k) \geq c_0(k, q)$.

The Structure of the Proof

The proof **relies on ...**

① Structural Results:

- Drudge (1998): $f(x) = x_P$ locally $\rightarrow f(x) = x_P$ globally.
- Drudge (1998): $f(x) = x_P + x_H$ locally $\rightarrow f(x) = x_P + x_H$ globally.
- Metsch (2010): f not trivial \rightarrow far away from trivial.

② Ramsey for vector spaces: Graham, Leeb, Rothschild (1972).

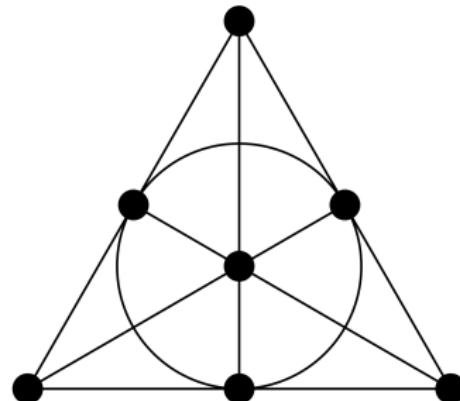
③ The case $k = 2$ suffices: Filmus, Ih. (2019).

My key insight was that we can use **Ramsey theory**.

Coloring the Fano Plane

Points of Fano plane = 1-spaces of \mathbb{F}_2^3 .

Lines of Fano plane = 2-spaces of \mathbb{F}_2^3 .



Question

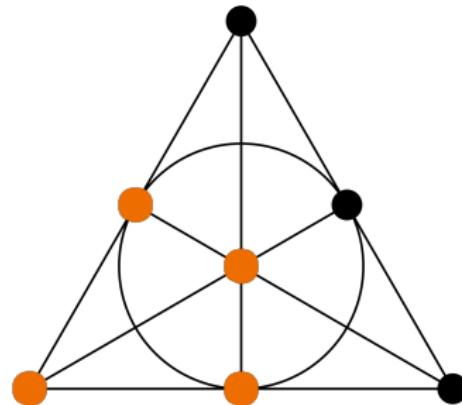
Can we color the points of the Fano plane black/orange with **no monochromatic line**?

Cf. Ex. 14.1.4 in *Discrete Mathematics: Elementary and Beyond* by L. Lovász, J. Pelikán, and K. Vesztergombi.

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No! This shows $R_2(2; 2) = 3$.

A formal definition of $R_q(s; m)$ follows on the **next slide**.

Ramsey Theory for Vector Spaces

Definition

A **m-coloring** of \mathbb{F}_q^n is a coloring of the 1-spaces of \mathbb{F}_q^n with m colors. The number $R_q(s; m)$ denotes the smallest integer n such that any m -coloring of \mathbb{F}_q^n possesses a monochromatic s -space.

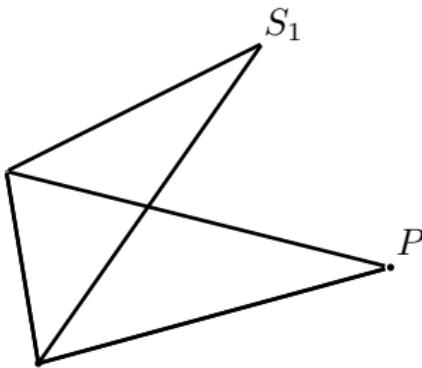
Theorem (Graham, Leeb, Rothschild (1972))

*The number $R_q(s; m)$ is **finite**.*

Theorem (Graham, Leeb, Rothschild (1972))

Analogous result for affine spaces.

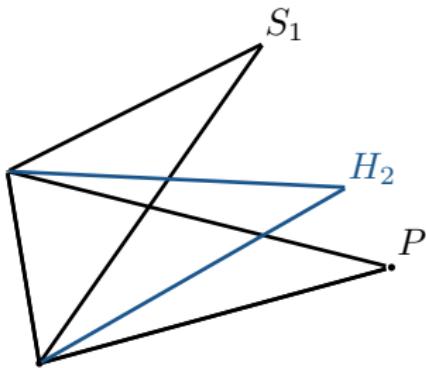
Applying Ramsey



How to apply **Ramsey**? Example for $q = 2$.

- ① Fix a 1-space P .
- ② **Goal:** the coefficient of x_P .
- ③ Say, **only 896 coefficients** can occur!
- ④ $n \geq R_2(s_1; 896)$: monochromatic s_1 -space S_1 .

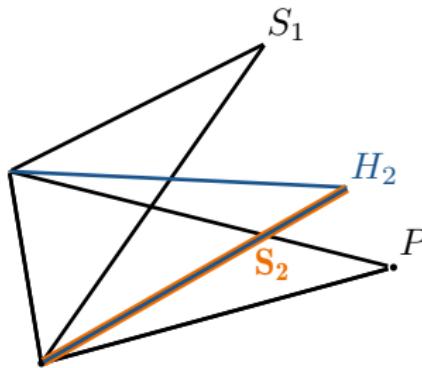
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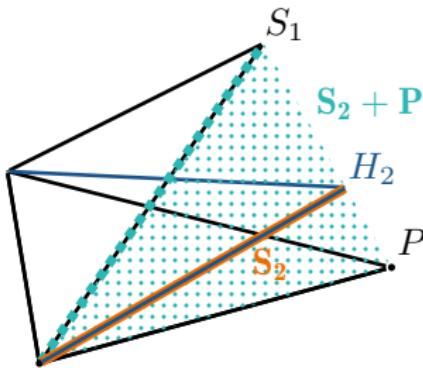
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- ⑤ $s_1 \geq R_2(s_2; 896)$: monochromatic affine s_2 -sp. S_2 in $H_2 \cap (S_1 + P)$.

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- ⑥ $s_2 \geq R_2(2; 896)$: monochromatic affine 2-space in $(S_1 \cap S_2) + P$.

Limited Weights

Ramsey gives us that any P has one of the coefficients

$$\left\{ -\frac{q}{q+1}, -\frac{q-1}{q}, -\frac{1}{q^2+q}, 0, \frac{1}{q}, \frac{1}{q+1}, 1 - \frac{1}{q^2+q}, 1 \right\}.$$

Proposition

If **all coefficients** are in $[-1, -\frac{q-1}{q+1}) \cup \{-\frac{1}{q^2+q}, 0, \frac{1}{q+1}, \frac{1}{q}\} \cup (\frac{q}{q+1}, 1]$, then f or $1 - f$ is one of $0, x_P, x_H, x_P + x_H$.

Proof.

- Drudge (1998),
- Metsch (2010),
- Some easy calculations.



What is $c_0(k, q)$?

Other arguments: $c_0(2, 2) = 2$.

Best known vector space Ramsey bound (I think):

Theorem (Frederickson, Yepremyan (2024, simplified))

$$R_2(s; m) \leq 2 \uparrow\uparrow ms.$$

⁵Also, I did not check the estimate below too carefully.

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Ignoring the difference between affine/projective, this gives⁵

$$\begin{aligned} 2 &= c_0(2, 2) \leq R_2(R_2(R_2(2; 896); 896); 896) - 2 \\ &\leq 2 \uparrow\uparrow (896 \cdot (2 \uparrow\uparrow (896 \cdot (2 \uparrow\uparrow 896 \cdot 2)))) - 2 \gg 2. \end{aligned}$$

Question

Does $c_0(2, q)$ grow in q ?

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Recent Breakthrough in Complexity Theory

The **Unique Games Conjecture** claims that it is impossible to approximate many **NP-hard** problems in polynomial time.

Theorem (Khot, Minzer, Safra (2023, Annals of Mathematics))

Proof of the 2-to-2 Games Conjecture.^a

^aA slightly weakened Unique Games Conjecture.

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Proof of the 2-to-2 Games Conjecture.^a

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What they had to show:

Theorem (Khot, Minzer, Safra (2023, Annals of Mathematics))

*Let $\alpha \in (0, 1)$. There ex. $\epsilon > 0$ s.t. for sufficiently large k and sufficiently large n : If f on k -spaces in \mathbb{F}_2^n **significant mass on low degree** (measured by α), then there ex. A of const. dim. and B of const. codim. with*

$$|\{x \in f : A \subseteq x \subseteq B\}| \geq \epsilon |\{x \text{ } \mathbf{k\text{-}space} : A \subseteq x \subseteq B\}|.$$

Think of $\dim(A) = 1$ and $\dim(B) = n$. Then $f = A^+$ is example.

Two-Intersecting Sets

Problem: Pick a set of 1-spaces \mathcal{P} in \mathbb{F}_q^n such that $|L \cap \mathcal{P}| \in \{a, b\}$ for any k -space L .
(two-intersecting set)

There are many examples for $(n, k) = (3, 2)$, e.g., **hyperovals**.

Always: take a 1-space or a hyperplane. (trivial examples)

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Theorem (Tallini Scafati (1976, simplified))

*For $(n, k) = (4, 2)$, if there is a non-trivial two-intersecting set, then q is an **odd square**.*

First open case is $q = 9$.

Theorem (Ih. (2024, AMS Proceedings))

*For k fixed and n sufficiently large, **all** two-intersecting sets are **trivial**.*

Future Work & Further Reading

Problem (Updated)

Investigate the behavior of $c_0(k, q)$. Does it grow in q ?

Problem (Nisan-Szegedy (Vector Spaces))

*On **how many variables** can a Boolean degree d function depend?*

Problem

*Can we improve the bounds for the **Ramsey number** $R_q(s; m)$?*

Reference for everything: Ih., A Survey of Cameron-Liebler Sets and Low Degree Boolean Functions in Grassmann Graphs, Summit280, Bolyai Society Mathematical Studies.



Thank you for your attention!