Sequentially Cohen-Macaulay binomial edge ideals of graphs

Francesco Romeo

University of Cassino and Southern Lazio, Italy

5th Pythagorean Conference – June 2, 2025

Section 1

Introduction

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Binomial edge ideal on G

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Neither y_2 nor $x_1 y_3 - x_3 y_1 \in J_G$. S/J_G is a domain if and only if G is a complete graph, that is, J_G is a determinantal ideal induced by the 2 by 2 minors of the 2 by n matrix

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}$$

Cut vertices and cutsets

The following is central in the study of binomial edge ideals

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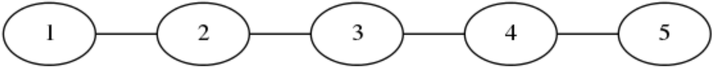

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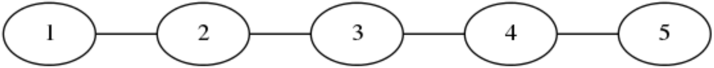

Its cutsets are

$$\emptyset, \{2\}, \{3\}, \{4\}, \{2, 4\}.$$

A bit of Commutative Algebra...

Let $S = K[x_1, \dots, x_n]$ a polynomial ring over a field K with a standard graduation, I homogeneous ideal, $\mathfrak{m}$ the maximal graded ideal of $R = S/I$.
Let $I = \bigcap_i^r Q_i$ be the primary decomposition of I and
$\operatorname{Ass} R = \{P_1, \dots, P_r\}$ the ideals $P_i = \sqrt{Q_i}$, $i = 1, \dots, r$.
We recall that

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2. An ideal is called unmixed if the prime ideals in $\operatorname{Ass} R$ have the same height;
3. $\theta_1, \theta_2, \ldots, \theta_r \in \mathfrak{m} \subset R$ is a homogeneous regular sequence if θ_i is a non zero-divisor of $R/(\theta_1, \ldots, \theta_{i-1})R$ for all i;

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4. $\operatorname{depth} R =$ maximum length between the lengths of regular sequences over R;
5. A ring is said Cohen-Macaulay if $\dim R = \operatorname{depth} R$.

A bit of Commutative Algebra...

Let M be a finitely generated graded module over R. The module M is called sequentially Cohen-Macaulay if there exists a finite filtration with graded submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that the following two conditions are fulfilled:

1. M_i / M_{i-1} is a Cohen-Macaulay module for $1 \leq i \leq r$;
2. $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$.

Goodarzi filtration of ideals

We recall a characterization of sequentially Cohen–Macaulay homogeneous ideals due to Goodarzi [G]
Let $I \subset S$ be a homogeneous ideal, with $d = \dim S/I$, and let $I = \bigcap_{j=1}^{r} Q_j$ be the minimal primary decomposition of I. For all $1 \leq j \leq r$, let $P_j = \sqrt{Q_j}$ be the radical of Q_j. For all $-1 \leq i \leq d$, denote

$$I^{<i>} = \bigcap_{\dim S/P_j > i} Q_j,$$

where $I^{<-1>} = I$ and $I^{<d>} = S$.

Proposition

Let $I \subset S$ be a homogeneous ideal and suppose $d = \dim S/I$. Then, S/I is sequentially Cohen-Macaulay if and only if

$$\operatorname{depth} S/I^{<i>} \geq i+1, \qquad \text{for all } 0 \leq i < d.$$

Primary decomposition of J_G

In [HHHKR] the authors observe that the binomial edge ideal is radical. Moreover its primary decomposition is

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$$P_T(G) = (\bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(T)}}),$$

$P_T(G)$ is prime and its generators $J_{\tilde{G}_{c(T)}}$ are the binomial edge ideals of complete graphs on the vertices of the connected components induced by the cutset T, $G_1, \dots, G_{c(T)}$,

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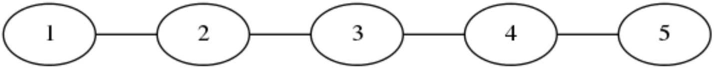
$$\dim S/P_T = n - |T| + c(T)$$

An example



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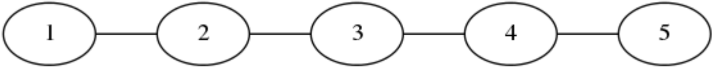

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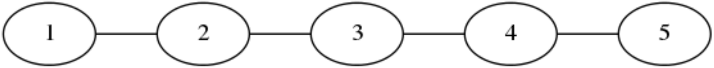

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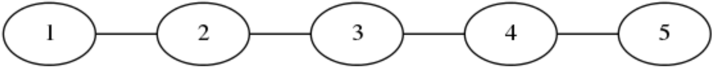

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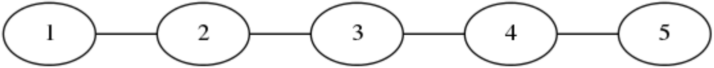

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Goodarzi + Binomial edge ideals

For a graph G set $\mathcal{D}(G) = \{\dim S/P_T \ : \ T \in \mathcal{C}(G)\}$.

Lemma (LR_)

Let G be a graph and let $\mathcal{D}(G) = \{d_1, \ldots, d_\ell\}$, with $d_1 < \ldots < d_\ell$. Then, S/J_G is Sequentially Cohen-Macaulay if and only if

$$\operatorname{depth} S/J_G^{<d_j-1>} = d_j, \qquad \textit{for all } j = 1, \ldots, \ell.$$

Hence, we only look at the d_i that define "depth layers".
Morover, d_ℓ is the Krull dimension of S/J_G, and we set $m(G) = d_1$.

Connectivity

A graph G on n vertices is *ℓ-vertex-connected* (or simply ℓ-connected) if $\ell < n$ and for every subset $T \subset V(G)$ of vertices such that $|T| < \ell$, the graph $G \setminus T$ is connected. The *vertex-connectivity* (or simply *connectivity*) of G, denoted by $\kappa(G)$, is the maximum integer ℓ such that G is ℓ-vertex-connected.

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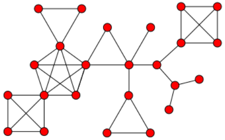

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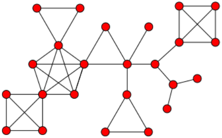

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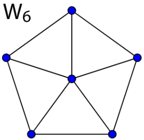


Connectivity and a bound on the depth

A nice result that relates depth S/J_G an $\kappa(G)$ has been obtained in

BB A. Banerjee, L. Núñez-Betancourt, "Graph connectivity and binomial edge ideals". Proc. Am. Math. Soc. , 487–499, 2017.

that is:

Theorem (BB)

Let G be a non–complete, connected graph on n vertices and J_G the corresponding binomial edge ideal. Then,

$$\operatorname{depth} S/J_G \leq n - \kappa(G) + 2,$$

where $\kappa(G)$ is the connectivity of the graph G.

Section 2

SCMness of Binomial Edge Ideals

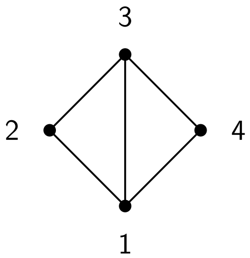
Condition for SCMness

Lemma (LR_)

Let G be a non–complete, connected graph on n vertices. Let $T \in \mathcal{C}(G)$ such that $\dim S/P_T = m(G)$. If J_G is SCM, then

$$\kappa(G) - |T| + c(T) \leq 2,$$

where $\kappa(G)$ is the connectivity of the graph G. In particular, if $T = \emptyset$, then G has a cutpoint.

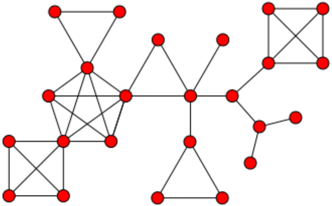


$k(G) = 2$
$\mathcal{C}(G) = \{\emptyset, \{1,3\}\}$

- $\dim P_\emptyset = 4 - 0 + 1 = 5$
- $\dim P_{\{1,3\}} = 4 - 2 + 2 = 4$

Block graphs

A *block* of G is a connected subgraph of G that has no cutpoints, which is maximal with respect to this property.
A *block graph* is a graph in which every block is a complete graph.

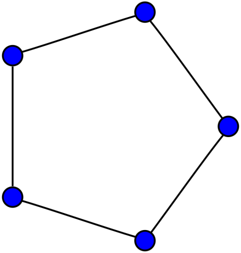


Theorem (LR_)

Any block graph is Sequentially Cohen-Macaulay.

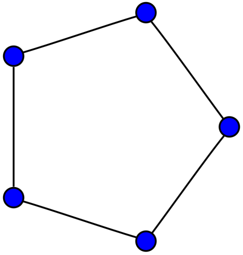
Cycles and wheels

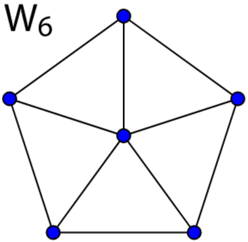
We start observing that all non empty cutsets of a cycle are sets of non adjacent vertices, and $c(T) = |T|$.



Cycles and wheels

We start observing that all non empty cutsets of a cycle are sets of non adjacent vertices, and $c(T) = |T|$.
Moreover, the non empty cutsets of a wheel are the cutsets of the cycle adding the central vertex, and $c(T) = |T| - 1$.





Cycles and wheels are SCM

Proposition (LR_)

Let C_n be a cycle with n vertices. Then, J_{C_n} is SCM.

Proof.

By the previous observation we have that $\mathcal{D}(G) = \{n, n+1\}$ Our aim is to apply Lemma to prove that S/J_{C_n} is SCM. That is S/J_{C_n} is SCM if

$$\operatorname{depth} S/{J_{C_n}}^{<n-1>} = n$$

and depth $S/{J_{C_n}}^{<n>} = n+1$. From ${J_{C_n}}^{<n-1>} = J_{C_n}$ and $S/{J_{C_n}}^{<n>} = P_\emptyset$, the assertion follows. □

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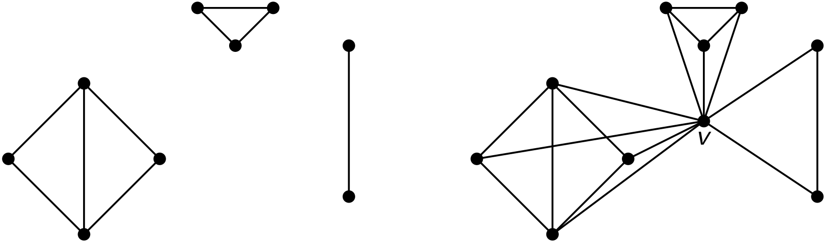
The same proof works for wheels, too!

Cone on multiple components



Theorem (LR_)

Let $H = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_r$ be a graph, where G_i is a connected graph for each $i = 1, \ldots, r$, with $r \geq 2$. Let v be a vertex such that $v \notin V(H)$ and $G = Cone(v, H)$. Then, J_G is SCM if and only if $J_{G_1}, \ldots, J_{G_r}$ are SCM.

Cone on a single component

Figure: A Sequentially Cohen-Macaulay graph G such that Cone(v, G) is not Sequentially Cohen-Macaulay

Theorem

Let H be a connected graph on vertices $[n-1]$ and let $G =$Cone(v, H), $R_H = \mathbb{K}[x_1, y_1, \dots, x_{n-1}, y_{n-1}]$ and $R = R_H[x_v, y_v]$. The following are equivalent

1. *J_G is Sequentially Cohen-Macaulay;*
2. *J_H is Sequentially Cohen-Macaulay and* $\dim R_H/J_H \leq n$

Section 3

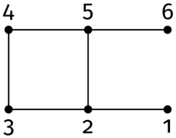
Open Questions and Remarks

Accessible graphs

BB [BMS] D. Bolognini, A. Macchia, F. Strazzanti, *Cohen-Macaulay binomial edge ideals and accessible graphs*, J. Algebr. Combin, 1139–1170, 2022.

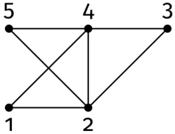
A graph G is called *accessible* if

1. J_G is unmixed
2. The set of cutsets is an *accessible set system*, i.e., for every non-empty cutset S, there exists $s \in S$ such that $S \setminus \{s\}$ is a cutset.



$\mathcal{C}(H) = \{\emptyset, \{2\}, \{5\}, \{2,5\}, \{2,4\}, \{3,5\}\}$

H accessible



$\mathcal{C}(G) = \{\emptyset, \{2,4\}\}$

G not accessible

SCM graphs Vs Accessible graphs

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Unfortunately, cycles and wheels are SCM but not "accessible".

1. Is it possible to extend the "accessible notion" to give a combinatorial interpretation to SCMness?
2. Find other families of SCM graphs that are accessible.

W.r.t. (2), by a Lemma of the first part, we have that if $\dim S/P_T = \mathrm{m}(G)$ with $T = \emptyset$, and J_G is SCM then G has a cutpoint. Using inductively this property one can find SCM families.

What are the SCM Bipartite graphs?

What we know:

SZ P. Schenzel, S. Zafar, *Algebraic properties of the binomial edge ideal of a complete bipartite graph*, Univ. "Ovidius" Constanța Ser. Mat. **22**(2), 217–237, 2014.

has been characterized the complete bipartite graphs that are SCM.
Moreover, in this talk we showed that:

1. All cycles are SCM. In particular the even ones.
2. All trees are block graphs, hence they are SCM.
3. The complete bipartite graphs not in 1. and 2. are not SCM by [SZ].

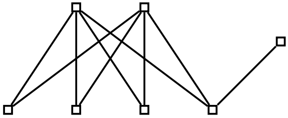

Figure: A non-SCM bipartite graph with cutpoint

THANKS FOR YOUR ATTENTION!!