Evaluation codes arising from symmetric polynomials

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2 June 2025

"INdAM - GNSAGA Project", codice CUP E53C24001950001

Outline

1. Background on evaluation codes
2. Datta-Johnsen Codes
3. Generalization of Datta-Johnsen Codes

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Background

- $\mathcal{F} \subseteq \mathbb{F}_q[X_1, \cdots, X_m]$
- $P_1, \cdots, P_n \in \mathbb{F}_q^m$

$$ev : \mathcal{F} \to \mathbb{F}_q^n \qquad f \mapsto (f(P_1), f(P_2), \ldots, f(P_n))$$

$ev(\mathcal{F}) = \{(f(P_1), \cdots, f(P_n)) | f \in \mathcal{F}\} \subseteq \mathbb{F}_q^n \longrightarrow$ evaluation code

- Reed-Solomon codes, Reed-Muller codes, monomial codes, Cartesian codes, toric codes

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Datta-Johnsen Codes

- $f \in \mathbb{F}_q[X_1, \ldots, X_m]$ is symmetric if

$$f(X_1, \ldots, X_m) = f\left(X_{\pi(1)}, \ldots, X_{\pi(m)}\right)$$

for any permutation π on the index set $\{1, \ldots, m\}$.

  - For $i = 1, \ldots, m$, the i-th elementary symmetric polynomial is

$$\sigma_m^i(X) = \sum_{1 \leq j_1 < \ldots < j_i \leq m} X_{j_1} \cdots X_{j_i}.$$

- $P = (a_1, \cdots, a_m) \in AG(m, q)$ is distinguished if $a_i \neq a_j$ whenever $i \neq j$

$AG_D(m, q) \longrightarrow$ distinguished points

M. Datta, T. Johnsen, *Codes from symmetric polynomials*, Des. Codes and Cryptogr. 91, 747–761 (2023)

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- $\Sigma_m \longrightarrow \mathbb{F}_q$ - linear combinations of $\sigma_m^1, \cdots, \sigma_m^m$
- $AG_D(m,q) = \{P_1, \cdots, P_n\}$

$$ev : \Sigma_m \to \mathbb{F}_q^n \qquad f \mapsto (f(P_1), f(P_2), \ldots, f(P_n))$$

$\mathcal{C}_m = ev(\Sigma_m) \longrightarrow$ *Datta-Johnsen Code*

Parameters : $n = \binom{q}{m} m!,\ k = m+1, d = (q-m)\binom{q-1}{m-1}(m-1)!$

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Reduced Datta-Johnsen Codes

$P_1, P_2 \in AG_D(m,q)$ are *equivalent* when they have the same coordinates but in a different order

$\mathcal{Q} = \{Q_1, \cdots, Q_N\}$ ordered set of representatives of equivalence classes of $AG_D(m,q)$

$$ev : \Sigma_m \to \mathbb{F}_q^N \qquad f \mapsto (f(Q_1), f(Q_2), \ldots, f(Q_N))$$

$\mathcal{C}'_m = ev(\Sigma_m) \longrightarrow$ *reduced Datta-Johnsen Code*

Parameters: $\quad N = \binom{q}{m},\ K = m+1,\ D = \binom{q}{m} - \binom{q-1}{m-1}$

Better rate!

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Generalization

$$\Phi_m : \mathbb{F}_q[X_1, \cdots, X_m] \to \mathbb{F}_q[X_1, \cdots, X_m]$$
$$f \quad \mapsto \Phi_m(f) = f(\sigma_m^1(x), \ldots, \sigma_m^m(x))$$

$Im(\Phi_m)$ consists of all symmetric polynomials in $\mathbb{F}_q[X_1, \cdots, X_m]$

- $\Sigma_{m,t} = \{f \in \mathbb{F}_q[X_1, \cdots, X_m] | deg(f) \leq t\}$

  or $\qquad \Longrightarrow \quad \Sigma$

- $\Sigma_{m,t}(r) \subseteq \Sigma_{m,t}$

$\Phi_m(\Sigma)$ is an $\mathbb{F}_q$ - subspace of symmetric polynomials

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- *Polynomials* $\longrightarrow \Phi_m(\Sigma)$
- *Points* $\longrightarrow AG_D(m,q) = \{P_1, \cdots, P_n\}$ distinguished points

$$ev : \Phi_m(\Sigma) \to \mathbb{F}_q^n \qquad f \mapsto (f(P_1), f(P_2), \ldots, f(P_n))$$

$ev(\Phi_m(\Sigma)) \longrightarrow$ ***generalized*** *Datta-Johnsen Code*

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Datta-Johnsen Codes $\longrightarrow t = 1$

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Datta-Johnsen Codes $\longrightarrow t = 1$

- $m = 2$ (q odd)

$$\varphi_2 \begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 x_2 \end{cases}$$

$q(q-1)$

$\Phi_2(\Sigma)$

$\downarrow$

generalized Datta-Johnsen Code

$\frac{1}{2}q(q-1)$

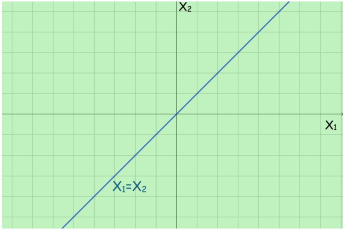
Σ

$\downarrow$

generalized reduced Datta-Johnsen Code

Our case

- $m = 2$ (q odd)



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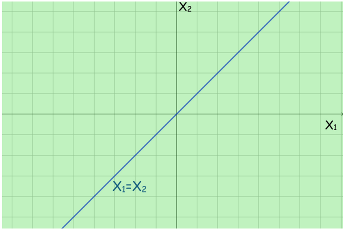

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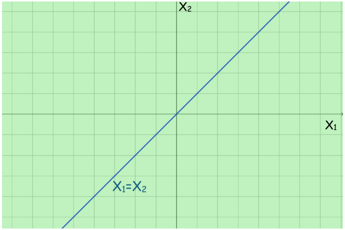

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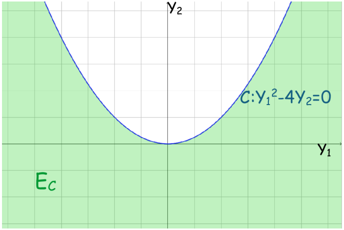

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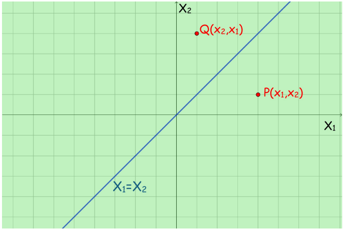

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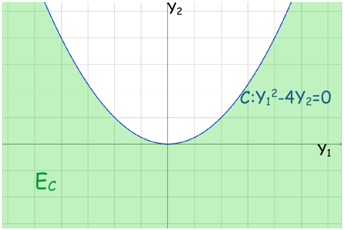

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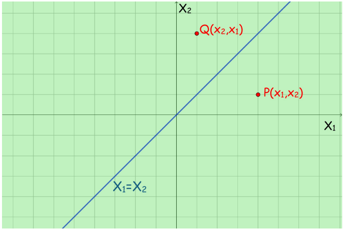

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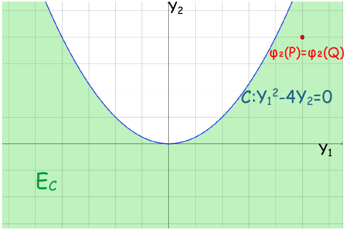
$$\varphi_2 \begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 x_2 \end{cases}$$



$\frac{1}{2}q(q-1)$

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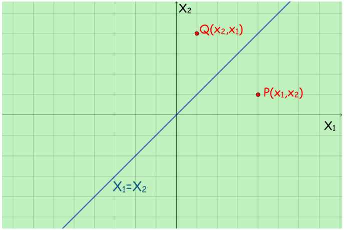

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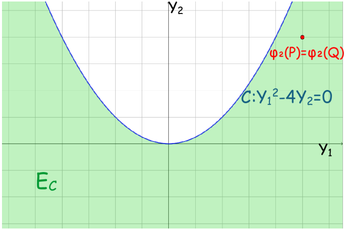

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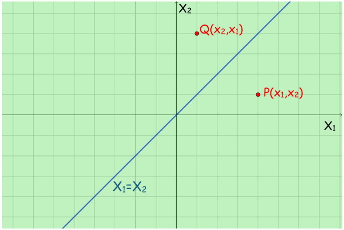

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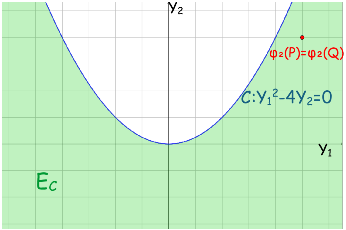

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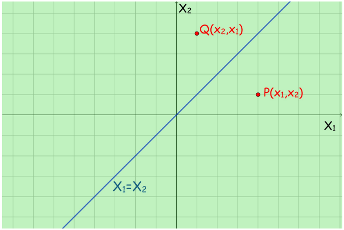

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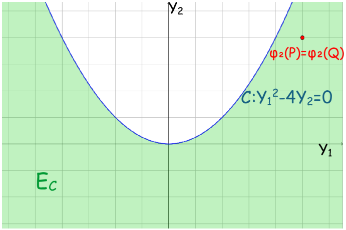

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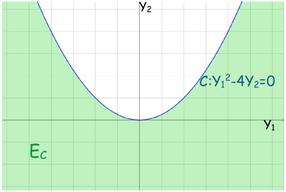

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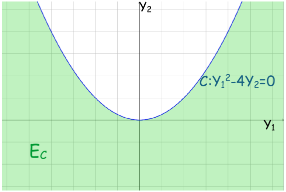

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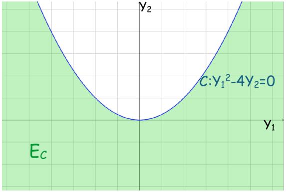

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Case of linear polynomials ($t = 1$)

$\Sigma = \Sigma_{2,1} = \{f \in \mathbb{F}_q[X_1, X_2] | deg(f) \leq 1\} \longrightarrow$ lines

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- *generalized reduced Datta-Johnsen Code*:
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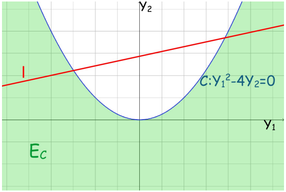

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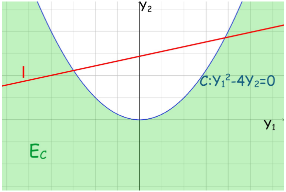

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Construction

$PG(2, q^3) \longrightarrow$
- the points of $PG(2, q)$;
- the points covered by lines of $PG(2, q)$;
- the set Λ of the remaining points.

$$\left.\begin{array}{l} P = (a : b : c) \in \Lambda \\ P_1 = (a^q : b^q : c^q) \\ P_2 = (a^{q^2} : b^{q^2} : c^{q^2}) \end{array}\right\} \longrightarrow \text{vertices of the triangle } PP_1P_2$$

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$\Sigma_2(3) = \{\mathcal{C}_\lambda | \lambda \in \mathbb{F}_{q^3} \setminus \{0\}\}$

$\mathcal{C}_\lambda : \lambda \ell_1(x,y,z)\ell_2(x,y,z) + \lambda^q \ell_2(x,y,z)\ell_3(x,y,z) + \lambda^{q^2} \ell_3(x,y,z)\ell_1(x,y,z) = 0$

- $\mathcal{C}_\lambda$ passes through P, P_1, P_2;
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- $|\Sigma_2(3)| = \frac{q^3-1}{q-1} = q^2 + q + 1 = |PG(2,q)|$;
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Case of quadratic polynomials ($t = 2$)

$(\mathcal{C}_\lambda \cap E_\mathcal{C})$?? $\longrightarrow$ it depends on P

$w_{max}(\mathcal{C}_2(3)') \leq \frac{1}{2}(q+1) + \sqrt{q} + 3$

- generalized reduced Datta-Johnsen Code:
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	q	Length	Dim	Min distance	Upper bound
$\mathcal{C}_2(3)$	5	10	3	**7**	**7**
	7	21	3	**16**	**17**
	9	36	3	**30**	**31**
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$(\mathcal{C}_\lambda \cap E_{\mathcal{C}})?? \longrightarrow$ it depends on P

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$$\left.\begin{array}{l} P = (a : b : c) \in PG(2, q^2) \setminus PG(2, q) \\ P_1 = (a^q : b^q : c^q) \\ P_2 \in PG(2, q) \text{ internal to the parabola } \mathcal{C} \end{array}\right\} \longrightarrow \text{vertices of the triangle } PP_1P_2$$

$\Sigma_2(3) \longrightarrow$ conics in $PG(2, q^2)$ passing through P, P_1, P_2 defined over $\mathbb{F}_q$

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The above idea doesn't work!!

$\Sigma_3(4) = \{\mathcal{C}_{\lambda,\delta} | \lambda \in \mathbb{F}_{q^3}, \delta \in \mathbb{F}_q, (\lambda, \delta) \neq (0,0)\}$

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