# QUADRATIC SETS AND $(t \mod q)$-ARCS IN $\mathrm{PG}(r, q)$

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– 5th Pytagorean Conference, Kalamata, Greece, June 1-6, 2025 –

# 1. $(t \mod q)$-Arcs

$\diamond$ A multiset in $\mathrm{PG}(r,q)$ is a mapping

$$\mathcal{K} : \begin{cases} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{cases}$$

$\diamond$ $\mathcal{K}(P)$ – multiplicity of the point $P$.

$\diamond$ $\mathcal{Q} \subset \mathcal{P}$: $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$ – multiplicity of the set $\mathcal{Q}$.

$\diamond$ $\mathcal{K}(\mathcal{P})$ – the cardinality of $\mathcal{K}$.

**Definition.** $(n,w)$-arc in $\mathrm{PG}(r,q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane $H$: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane $H_0$: $\mathcal{K}(H_0) = w$.

**Definition.** $(n,w)$-blocking set in $\mathrm{PG}(r,q)$

(or $(n,w)$-minihyper): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane $H$: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane $H_0$: $\mathcal{K}(H_0) = w$.

Definition. Let $t < q$ be a non-negative integer.

An arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called a **$(t \mod q)$-arc** if every subspace $S$ of positive dimension has multiplicity $\mathcal{K}(S) \equiv t \pmod{q}$.

If in addition, every point $P$ has multiplicity at most $t$, i.e. $\mathcal{K}(P) \leq t$; the $\mathcal{K}$ is called a **strong $(t \mod q)$-arc**.

Remark. It is enough to require the congruence in the definition only for the the subspeces of dimension 1 (i.e. for the lines).

**Theorem A.** Let $t_1 < q$ and $t_2 < q$ be positive integers. The sum of a $(t_1 \mod q)$-arc and a $(t_2 \mod q)$-arc in $\mathrm{PG}(r, q)$ is a $(t \mod q)$-arc with $t = t_1 + t_2 \pmod q$.

In particular, the sum of $t$ hyperplanes in $\mathrm{PG}(r, q)$ is a strong $(t \mod q)$-arc.

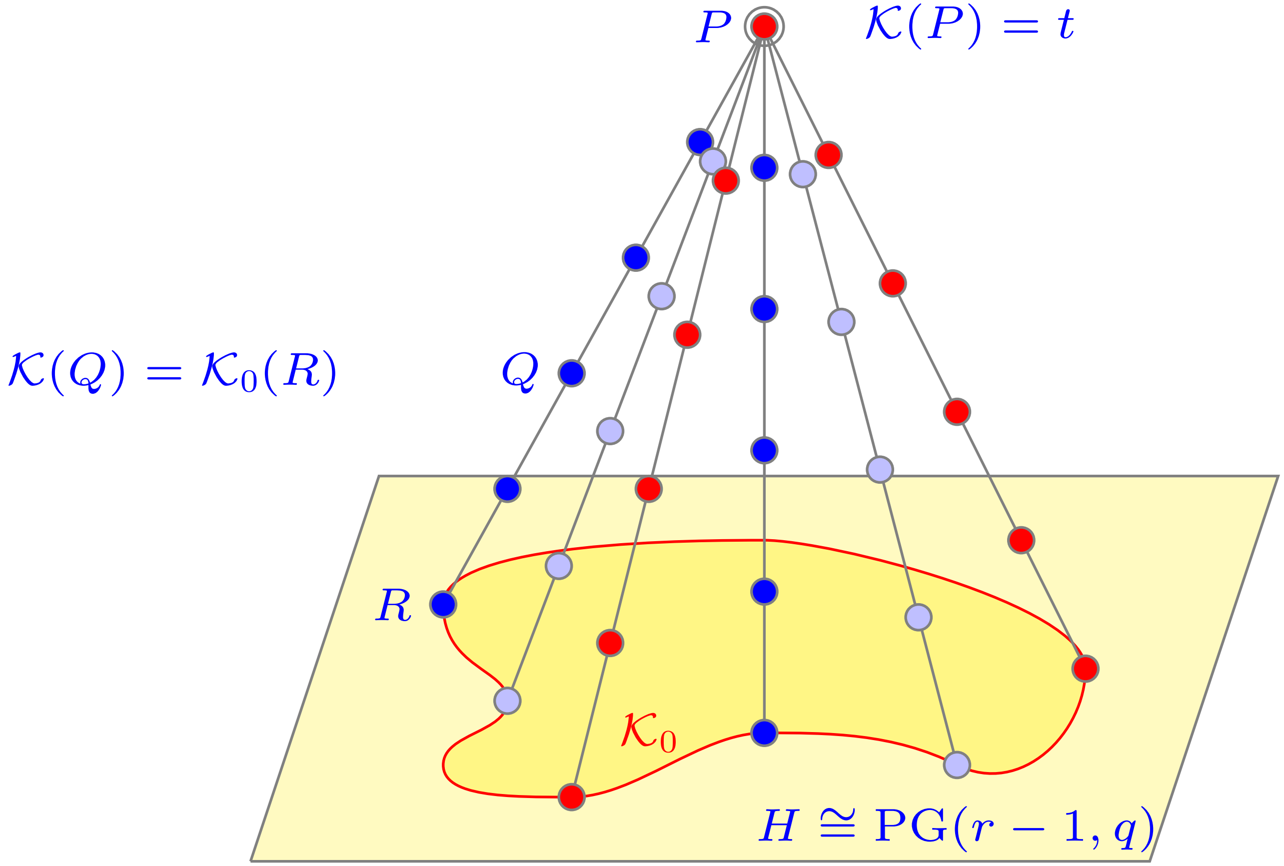
**Theorem B.** Let $\mathcal{K}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$. of $\Sigma = \mathrm{PG}(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{K}$ in $\Sigma$ as follows:

– $\mathcal{K}(P) = t$;

– for each point $Q \neq P$: $\mathcal{K}(Q) = \mathcal{K}_0(R)$ where $R = \langle P, Q \rangle \cap H$.

Then the arc $\mathcal{K}$ is a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ of size $q|\mathcal{K}_0| + t$.

**Definition**. $(t \mod q)$-arcs obtained by Theorem D are called **lifted arcs**.


$P$
$\mathcal{K}(P) = t$
$\mathcal{K}(Q) = \mathcal{K}_0(R)$
$Q$
$R$
$\mathcal{K}_0$
$H \cong \mathrm{PG}(r-1, q)$

**Theorem C.** A strong $(t \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(2, q)$ of cardinality $mq + t$ exists if and only if there exists an $((m-t)q + m, m-t)$-blocking set $\mathcal{B}$ with line multiplicities contained in $\{m-t, m-t+1, \ldots, m\}$.

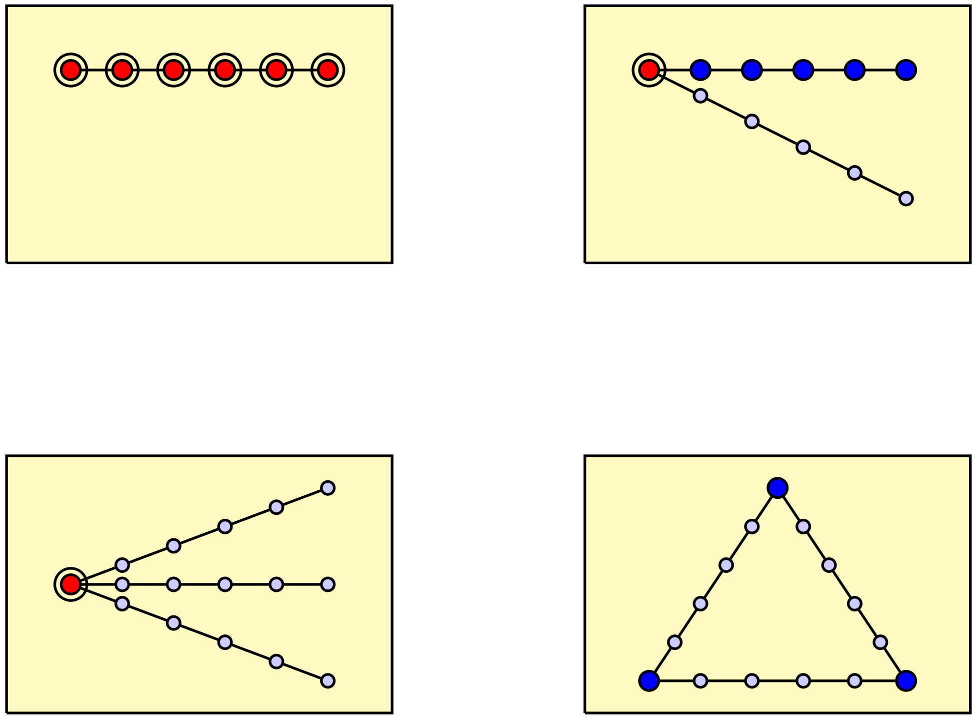
| | | |
|---|---|---|
| $(1 \mod q)$ | $\mathrm{PG}(r, q)$ | a hyperplane |
| $(2 \mod q)$ | $\mathrm{PG}(2, q)$, $q \geq 5$ | lifted from a $2, q+2$, or $2q+2$-line, or an oval + a tangent $+2\times$the internal points |
| | $\mathrm{PG}(r, q)$, $r \geq 3$ | lifted from a $(2 \mod q)$-arc in $\mathrm{PG}(r, q)$ |
| $(3 \mod 5)$ | $\mathrm{PG}(2, 5)$ | $185$ arcs |
| | $\mathrm{PG}(3, 5)$ | lifted and three sporadic $(3 \mod 5)$-arcs of sizes $128, 143$, and $168$ |
| | $\mathrm{PG}(r, 5)$, $r \geq 4$ | lifted and ... |

## 2. Strong $(3 \mod 5)$-Arcs in $\mathrm{PG}(2,5)$

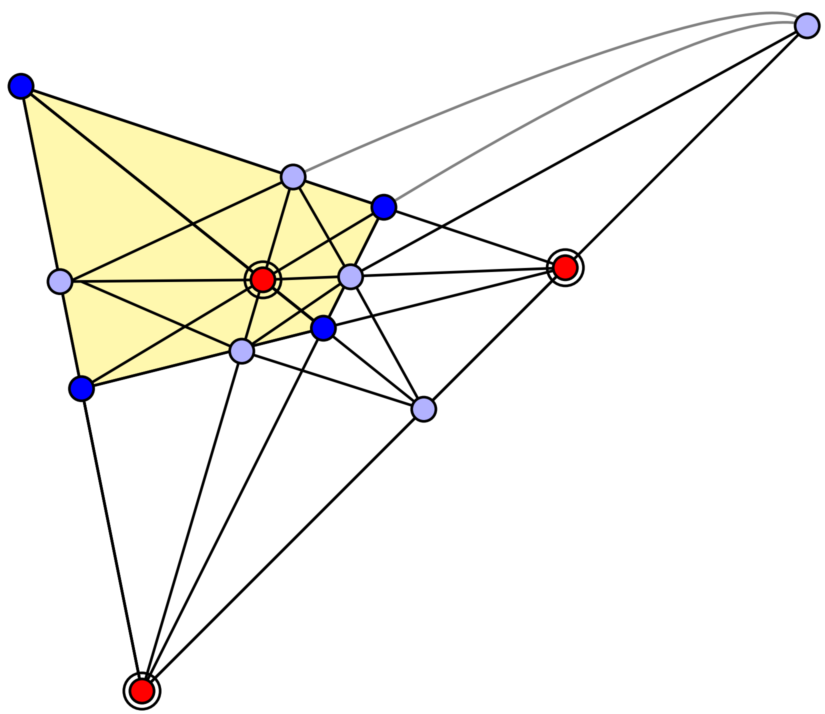
| $\lvert\mathcal{K}\rvert$ | BS | # arcs | $\lvert\mathcal{K}\rvert$ | BS | # arcs |
|---|---|---|---|---|---|
| 18 | $(3,0)$ | 4 | 48 | $(39,6)$ | 49 |
| 23 | $(9,1)$ | 1 | 53 | $(45,7)$ | 17 |
| 28 | $(15,2)$ | 1 | 58 | $(51,8)$ | 11 |
| 33 | $(21,3)$ | 10 | 63 | $(57,9)$ | 9 |
| 38 | $(27,4)$ | 23 | 68 | $(63,10)$ | 6 |
| 43 | $(33,5)$ | 53 | 93 | $(93,15)$ | 1 |

- Ivan Landjev & Assia Rousseva (computerfree)
- Sascha Kurz (computer search)

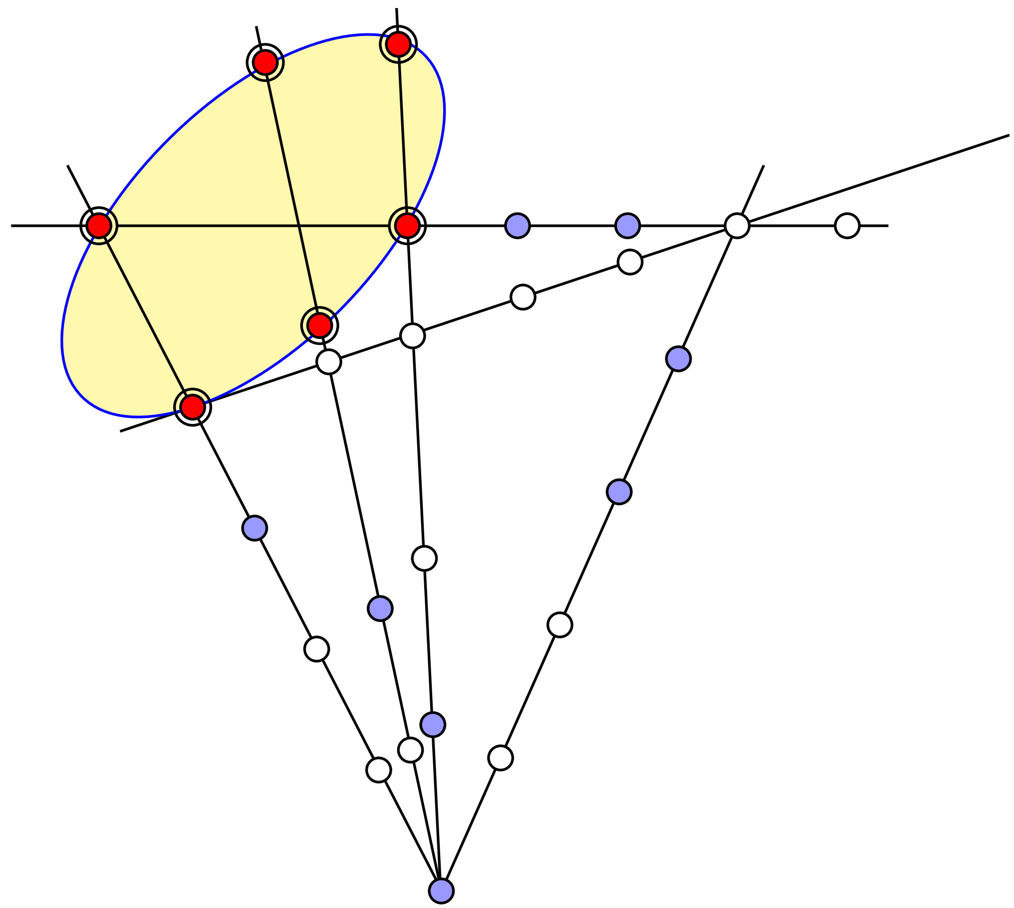
# $(18, \{3, 8, 13, 18\})$-arcs



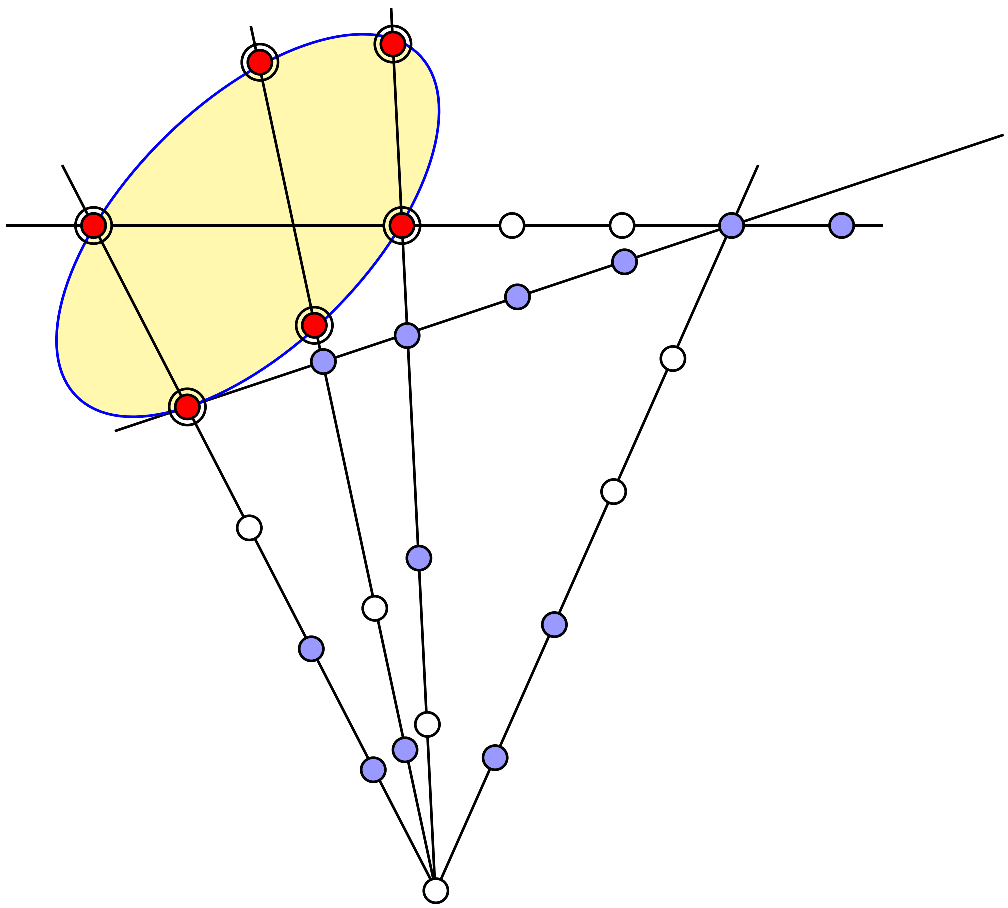
# $(23, \{3, 8\})$-arc in $\mathrm{PG}(2, 5)$



## $(28, \{3, 8\})$-arc in $\mathrm{PG}(2, 5)$



## $(33, \{3, 8\})$-arc in $\mathrm{PG}(2, 5)$



# 3. $(3 \mod 5)$-Arcs in $\mathrm{PG}(3,5)$

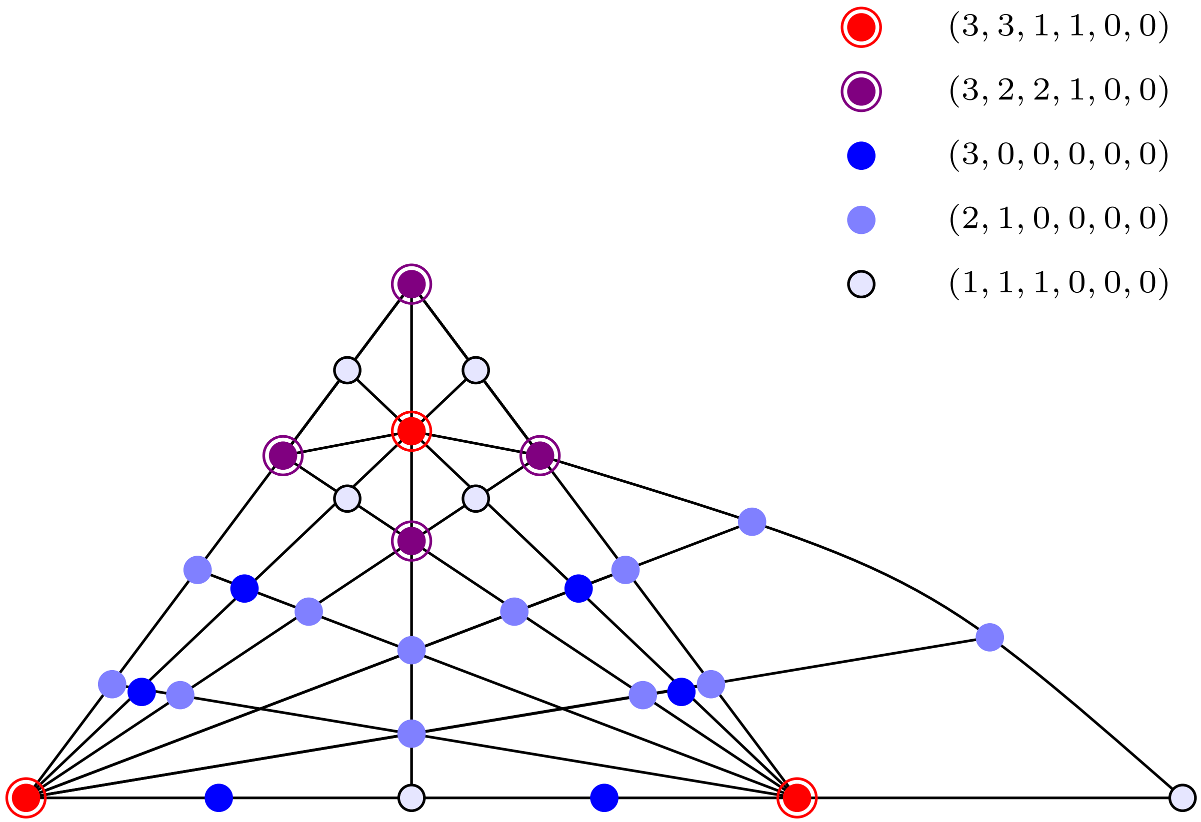
**Theorem D.** Let $\mathcal{K}$ be a strong $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ that is neither lifted nor contains a full hyperplane in its support. Then $|\mathcal{K}| \in \{128, 143, 168\}$ and in each case the corresponding arc is unique up to isomorphism.

## 4. The 128-Arc in $\mathrm{PG}(3,5)$

**Lemma.** Let $\mathcal{K}$ be a strong $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ of cardinality 128. Let $\varphi$ be the projection from an arbitrary 0-point in $\mathrm{PG}(3,5)$. Then the arc $\mathcal{K}^{\varphi}$ is unique up to isomorphism and has the structure described below.

- A 0-point is incident only with 3- and 8-lines.
- An 8-line with a 0-point is of type $(3,3,1,1,0,0)$ or $(3,2,2,1,0,0)$.

# Projection of a 128-arc from a 0-point



Each 0-point is incident with:

three 8-lines of type $(3, 3, 1, 1, 0, 0)$,
four 8-lines of type $(3, 2, 2, 1, 0, 0)$,
six 3-lines of type $(3, 0, 0, 0, 0, 0)$,
twelve 3-lines of type $(2, 1, 0, 0, 0, 0)$,
six 3-lines of type $(1, 1, 1, 0, 0, 0)$

This implies that

$$\lambda_3 = 16, \lambda_2 = 20, \lambda_1 = 40, \lambda_0 = 80.$$

$$a_{33} = 40, a_{28} = 16, a_{23} = 80, a_{18} = 20.$$

The 2-points form a 20-cap $C$ with spectrum:

$$a_6(C) = 40, a_4(C) = 80, a_3(C) = 20, a_0(C) = 16.$$

This cap is **not** extendable to the elliptic quadric. In such case it would have (at least 20) tangent planes.

Hence the 20-cap on the 2-points in $\mathrm{PG}(3,5)$ is isomorphic to one of the two maximal 20-caps found by Abatangelo, Korchmaros and Larato. We denote these two caps by $K_1$ and $K_2$.

Consider the complete cap $K_1$. The collineation group $G$ of $K_1$ is a semidirect product of an elementary abelean group of order 16 and a group isomorphic to $S_5$. Hence $|G| = 16 \cdot 120 = 1920$.

The action of $G$ on $\mathrm{PG}(3,5)$ gives four orbits on points, denoted $O_1^P, \dots, O_4^P$ and six orbit on lines, denoted $O_1^L, \dots, O_6^L$.

The respective sizes of these orbits are

$$|O_1^P| = 40, |O_2^P| = 80, |O_3^P| = 20, |O_4^P| = 16;$$

$$|O_1^L| = 160, |O_2^L| = 240, |O_3^L| = 30, |O_4^L| = 160, |O_5^L| = 120, |O_6^L| = 96.$$

The point-by-line orbit matrix $A = (a_{ij})_{4\times 6}$, where $a_{ij}$ is the number of the points from the $i$-th point orbit incident with any line from the $j$-th line orbit is the following

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 2 & 0 \\ 3 & 4 & 0 & 2 & 2 & 5 \\ 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

Let $w_i$ be the multiplicity of any point from $O_i^P$ and let $w = (w_1, w_2, w_3, w_4)$. In order to get a $(3 \mod 5)$-arc we should have

$$wA \equiv 3\boldsymbol{j} \pmod 5,$$

where $\boldsymbol{j}$ is the all-one vector, and $w_i \leq 3$ for all $i = 1, 2, 3$.

The set of all solutions is given by

$$w = \{(w_1, w_2, w_3, w_4) \mid w_i \in \{0, \dots 4\},$$
$$w_2 \equiv 1 - w_1 \pmod 5, w_3 \equiv 4 - 2w_1 \pmod 5, w_4 = 3\}. \quad (1)$$

Solutions: $w = (3, 3, 3, 3)$ and $w = (1, 0, 2, 3)$.

The second solution gives the desired $128$-arc.

# 5. The 143- and 168-Arc

Two strong non-lifted $(3 \mod 5)$-arcs in $\mathrm{PG}(3,5)$ were constructed by computer search. The respective spectra are:

$|\mathcal{F}_1| = 143$, $a_{18}(\mathcal{F}_1) = 26, a_{28}(\mathcal{F}_1) = 65, a_{33}(\mathcal{F}_1) = 65$;

$$\lambda_0(\mathcal{F}_1) = 65, \lambda_1(\mathcal{F}_1) = 65, \lambda_2(\mathcal{F}_1) = 0, \lambda_3(\mathcal{F}_1) = 26,$$

$|\operatorname{Aut}(\mathcal{F}_1)| = 62400$.

$|\mathcal{F}_2| = 168$, $a_{28}(\mathcal{F}_2) = 60, a_{33}(\mathcal{F}_2) = 60, a_{43}(\mathcal{F}_2) = 36$;

$$\lambda_0(\mathcal{F}_2) = 60, \lambda_1(\mathcal{F}_2) = 60, \lambda_2(\mathcal{F}_2) = 0, \lambda_3(\mathcal{F}_2) = 36.$$

$|\operatorname{Aut}(\mathcal{F}_2)| = 57600$.

There exist two quadrics in $\mathrm{PG}(3,5)$.

$$\mathcal{E}_3 = \{P(X_0, X_1, X_2, X_3) \mid X_0^2 + 2X_1^2 + X_2X_3 = 0, \} \tag{2}$$

$$\mathcal{H}_3 = \{P(X_0, X_1, X_2, X_3) \mid X_0X_1 + X_2X_3 = 0, \} \tag{3}$$

- $\mathcal{F}_1$: for a point $P(x_0, x_1, x_2, x_3)$ set

$$\mathcal{F}_1(P) = \begin{cases} 3 & \text{if } P \in \mathcal{E}_3, \\ 1 & \text{if } x_0^2 + 2x_1^2 + x_2x_3 \text{ is a square in } \mathbb{F}_5, \\ 0 & \text{if } x_0^2 + 2x_1^2 + x_2x_3 \text{ is a non-square in } \mathbb{F}_5. \end{cases} \tag{4}$$

- $\mathcal{F}_2$: for a point $P(x_0, x_1, x_2, x_3)$ set

$$\mathcal{F}_2(P) = \begin{cases} 3 & \text{if } P \in \mathcal{H}_3, \\ 1 & \text{if } x_0x_1 + x_2x_3 \text{ is a square in } \mathbb{F}_5, \\ 0 & \text{if } x_0x_1 + x_2x_3 \text{ is a non-square in } \mathbb{F}_5. \end{cases} \tag{5}$$

Let $F$ be a quadratic form in $r+1$ variables. Define an arc $\mathcal{F}$ in the following way:

$$\mathcal{F}(P(X)) = \begin{cases} \frac{q+1}{2} & \text{if } F(X) = 0, \\ 1 & \text{if } F(X) \text{ is a square/non-square,} \\ 0 & \text{if } F(X) \text{ is a non-square/square.} \end{cases}$$

This arc is a strong non-lifted $(t \mod q)$-arc with $t = \frac{q+1}{2}$.

Arcs obtained by this construction are called **quadratic** $(t \mod q)$-**arcs**.

# The case $q = 5$

**Theorem E.** Assume that every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r, 5)$, which does not contain a hyperplane in its support is lifted or obtained from a quadric. Then every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r+1, 5)$, is also lifted or a quadratic arc.

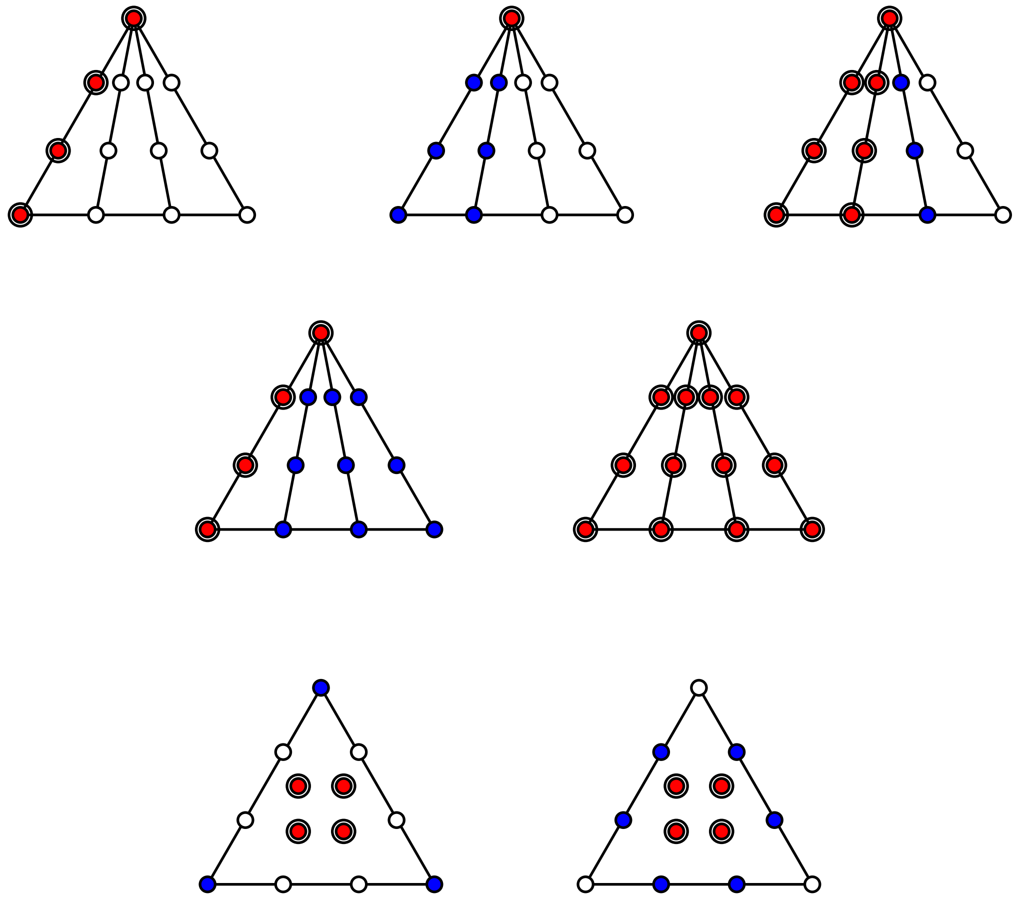
**Theorem F.** Every strong $(3 \mod 5)$-arc in $\mathrm{PG}(4, 5)$, which does not contain a hyperplane in its support is lifted or a quadratic arc.

**Corollary.** Every strong $(3 \mod 5)$-arc in $\mathrm{PG}(r, 5)$, $r \geq 4$, which does not contain a hyperplane in its support is lifted or a quadratic arc.

# The case $q = 3$

**Theorem G.** Assume that every strong $(2 \mod 3)$-arc in $\mathrm{PG}(r, 3)$ is lifted or obtained from a quadric. Then every strong $(2 \mod 3)$-arc in $\mathrm{PG}(r+1, 3)$, is also lifted or a quadratic arc.

**Theorem H.** Every strong $(2 \mod 3)$-arc in $\mathrm{PG}(2, 3)$, is lifted or a quadratic arc.



**Corollary.** Every strong $(2 \mod 3)$-arc in $\mathrm{PG}(r,3)$, $r \geq 2$, is lifted or a quadratic arc.