

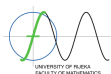
Self-orthogonal and LCD codes related to some combinatorial structures

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Preliminaries

Self-orthogonal codes from p -WSO designs

LCD codes from p -WSO designs

Self-orthogonal codes from digraphs

Codes

We will talk only about **linear codes**, i.e. subspaces of the ambient vector space over a field \mathbb{F}_q of order $q = p^l$, where p is prime.

A code C of length n and dimension k is denoted by $[n, k]$.

By $[n, k, d]_q$, we denote code C with minimal distance d over the field \mathbb{F}_q .

Specially, if $q = 2$, parameters of code C are denoted by $[n, k, d]$.

A **generator matrix** of a $[n, k]$ code C is a $k \times n$ matrix whose rows generate all the words of C .

The **dual code** of a code C is code C^\perp , $C^\perp = \{v \in (\mathbb{F}_q)^n \mid (v, c) = 0, \forall c \in C\}$.

A code is **self-orthogonal** if $C \subseteq C^\perp$. A code is **self-dual** if $C = C^\perp$.

A code is **LCD (Linear code with complementary dual)** if $C \cap C^\perp = \{0\}$.

A code C with generator matrix G is LCD code if and only if $\det(G \cdot G^T) \neq 0$.



Weakly self-orthogonal designs

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set \mathcal{P} , block set \mathcal{B} and incidence \mathcal{I} is called a $t - (v, k, \lambda)$ design, if \mathcal{P} contains v points, every block $B \in \mathcal{B}$ is incident with k points, and every t distinct points are incident with λ blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A design is weakly p -self-orthogonal (p -WSO) if all block intersection numbers gives the same residue modulo p .

A weakly p -self-orthogonal design is p -self-orthogonal if block intersection numbers and the block sizes are multiples of p .

Specially, weakly 2-self-orthogonal design is called weakly self-orthogonal (WSO) design, and 2-self-orthogonal design is called self-orthogonal.



Orbit matrices of a design

Let \mathcal{D} be a $1-(v, k, \lambda)$ design and G be an automorphism group of the design. Let $v_1 = |\mathcal{V}_1|, \dots, v_n = |\mathcal{V}_n|$ be the sizes of point orbits and $b_1 = |\mathcal{B}_1|, \dots, b_m = |\mathcal{B}_m|$ be the sizes of block orbits under the action of the group G . We define an **orbit matrix** as $m \times n$ matrix

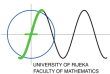
$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij} is the number of points of the orbit \mathcal{V}_j incident with a block of the orbit \mathcal{B}_i . It is easy to see that the matrix is well-defined and that $k = \sum_{j=1}^n a_{ij}$.

For $x \in \mathcal{B}_s$, by counting the incidence pairs (P, x') such that $x' \in \mathcal{B}_t$ and P is incident with the block x , we obtain

$$\sum_{x' \in \mathcal{B}_t} |x \cap x'| = \sum_{j=1}^m \frac{b_t}{v_j} a_{sj} a_{tj} = \frac{b_t}{v_j} O[s] \cdot O[t],$$

where $O[s]$ is the s -th row of the matrix O .



Let \mathcal{D} be a weakly p -self-orthogonal design such that

$$k \equiv a \pmod{p}$$

and

$$|B_i \cap B_j| \equiv d \pmod{p},$$

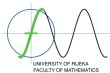
for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design \mathcal{D} .

Let G be an automorphism group of the design which acts on \mathcal{D} with n point orbits of length w and block orbits of length b_1, b_2, \dots, b_m , and let O be an orbit matrix of a design \mathcal{D} under the action of a group G .

Let $q = p^n$ be prime power and let \mathbb{F}_q be a finite field of order q . In \mathbb{F}_q , for $x \in \mathcal{B}_s$, $s \neq t$ it follows that

$$\frac{b_t}{w} O[s] \cdot O[t] = b_t d, \tag{1}$$

$$\frac{b_s}{w} O[s] \cdot O[s] = a + (b_s - 1)d. \tag{2}$$



Submatrices of orbit matrix of p -WSO design

Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q .

Let $\mathcal{D} 1-(v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} .

Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α .

$$O = \left[\begin{array}{cccc|cccc} & & & & a_{1,f_1+1} & a_{1,f_1+2} & \cdots & a_{1,f_1+n} \\ & & & & a_{2,f_1+1} & a_{2,f_1+2} & \cdots & a_{2,f_1+n} \\ & & & & \vdots & \vdots & \ddots & \vdots \\ \text{OM1} & & & & a_{f_2,f_1+1} & a_{f_2,f_1+2} & \cdots & a_{f_2,f_1+n} \\ \hline a_{f_2+1,1} & a_{f_2+1,2} & \cdots & a_{f_2+1,f_1} & & & & \\ a_{f_2+2,1} & a_{f_2+2,2} & \cdots & a_{f_2+2,f_1} & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ \text{OM2} & & & & & & & \\ a_{f_2+m,1} & a_{f_2+m,2} & \cdots & a_{f_2+m,f_1} & & & & \end{array} \right]$$



Self-orthogonal codes from p -WSO designs

Let \mathbb{F}_q be a finite field of order $q = p^l$, where p is a prime.

- ▶ Construction of SO codes obtained from incidence matrix of p -WSO designs, using suitable extensions of incidence matrix of a design.
- ▶ Construction of SO codes obtained from orbit matrix of p -WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w , using suitable extensions of orbit matrix.
- ▶ Construction of SO codes obtained from submatrices of orbit matrix of p -WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α , and with f_2 fixed blocks and m block orbits of length p^α , $1 \leq \alpha \leq n$.

SO codes from incidence matrix of WSO 1-designs

Theorem (V. Tonchev)

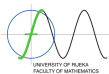
Let \mathcal{D} be weakly self-orthogonal design and let M be it's $b \times v$ incidence matrix.

1. If \mathcal{D} is a self-orthogonal design, then the matrix M generates a binary self-orthogonal code.
2. If \mathcal{D} is such that k is even and the block intersection numbers are odd, then the matrix $[I_b, M, \mathbf{1}]$ generates a binary self-orthogonal code.
3. If \mathcal{D} is such that k is odd and the block intersection numbers are even, then the matrix $[I_b, M]$ generates a binary self-orthogonal code.
4. If \mathcal{D} is such that k is odd and the block intersection numbers are odd, then the matrix $[M, \mathbf{1}]$ generates a binary self-orthogonal code.

SO codes from incidence matrix of p -WSO 1-designs

Let $q = p^l$ be prime power and \mathbb{F}_q a finite field of order q . Let \mathcal{D} be a weakly p -self-orthogonal design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design \mathcal{D} . Let M be its $b \times v$ incidence matrix.

1. If \mathcal{D} is p -self-orthogonal design, then M generates a self-orthogonal code over \mathbb{F}_q .
2. If $a = 0$ and $d \neq 0$, then the matrix $[\sqrt{d} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
3. If $a \neq 0$ and $d = 0$, then the matrix $[M, \sqrt{-a} \cdot I_b]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
4. If $a \neq 0$ and $d \neq 0$, there are two cases:
 - ▶ if $a = d$, then the matrix $[M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise, and
 - ▶ if $a \neq d$, then the matrix $[\sqrt{d-a} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in \mathbb{F}_q , and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.

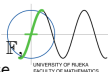


SO codes form orbit matrices of p -WSO 1-designs

Let $q = p^l$ be prime power and \mathbb{F}_q a finite field of order p .

Let \mathcal{D} be a weakly p -self-orthogonal 1-design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design \mathcal{D} , and let G be an automorphism group of the design which acts on \mathcal{D} with n point orbits of length w and m block orbits of length w and let O be the orbit matrix of \mathcal{D} under action of a group G .

- ▶ If $a = d$ we differ two cases.
 - a) If $p \mid w$, the linear code spanned by the rows of the matrix O is self-orthogonal code over the field \mathbb{F}_q .
 - b) If $p \nmid w$, the linear code spanned by the rows of the matrix $[O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-wd$ is a square in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
- ▶ If $a \neq d$, we differ three cases.
 - a) If $p \mid w$, the linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, O]$ is self-orthogonal code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $d-a$ kvadrat u polju \mathbb{F}_q i $\mathbb{F} = \mathbb{F}_{q^2}$ inače.
 - b) If $p \mid w-1$, the linear code spanned by the rows of the matrix $[\sqrt{wd-a} \cdot I_m, O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal $[m+n, m]$ code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $wd-a$ and $-wd$ are squares in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise. If $m = n$, the obtained code is self-dual.
 - c) If $p \nmid w$ i $q \nmid w-1$, the linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal $[m+n+1, m]$ code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $d-a$ and $-wd$ are squares in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.



SO codes obtained using submatrices of orbit matrix of p -WSO design

Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q .

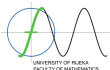
Let $\mathcal{D} 1-(v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} . Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α .

► If $a = d$, we differ two cases.

1. Linear code spanned by the rows of the matrix $[OM1, \sqrt{-a} \cdot \mathbf{1}]$ is self-orthogonal code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
2. Linear code spanned by the rows of the matrix $OM2$ is self-orthogonal code over the field \mathbb{F}_q .

► If $a \neq d$, we differ two cases.

1. Linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_{f_2}, OM1, \sqrt{-d} \cdot \mathbf{1}]$ is self-orthogonal $[f_2 + f_1 + 1, f_2]$ code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $d-a$ and $-d$ are squares in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
2. Linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, OM2]$ is self-orthogonal $[m+n, m]$ code over the field \mathbb{F} , where $\mathbb{F} = \mathbb{F}_q$ if $d-a$ is a square in the field \mathbb{F}_q and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise. If $m = n$, the obtained code is self-dual.



Some results...

Code	Design
$[6, 2, 4]^*$	$1-(22, 10, 10)$
$[10, 4, 4]^*$	$1-(110, 72, 36)$
$[12, 2, 8]^*$	$1-(132, 100, 100)$
$[12, 3, 6]^*$	$1-(132, 66, 6)$
$[12, 5, 4]^*$	$1-(132, 40, 40)$
$[12, 5, 4]^*$	$1-(132, 46, 46)$
$[12, 6, 4]^*$	$1-(66, 45, 45)$
$[12, 11, 2]^*$	$1-(132, 22, 2)$
$[15, 4, 8]^*$	$1-(165, 48, 48)$
$[24, 12, 8]^*$	$1-(132, 27, 27)$
$[16, 5, 8]^*$	$1-(165, 109, 109)$
$[31, 15, 8]^*$	$1-(165, 116, 116)$
$[9, 2, 6]_3^*$	$1-(30, 21, 7)$
$[6, 3, 2]^{**}$	$1-(22, 2, 1)$
$[10, 3, 4]^{**}$	$1-(66, 46, 46)$
$[14, 6, 4]^{**}$	$1-(110, 72, 72)$
$[64, 4, 32]^{**}$	$1-(132, 55, 5)$
$[30, 4, 18]_3^{**}$	$1-(30, 18, 3)$
$[96, 48, 16]^+$	$1-(144, 23, 23)$

Table: Optimal(*), near-optimal(**) and best known(+) SO codes

LCD codes from p -WSO designs

Let \mathbb{F}_q be a finite field of order $q = p^l$, where p is a prime.

- ▶ Construction of LCD codes obtained from incidence matrix of p -WSO designs, using suitable extensions of incidence matrix of a design.
- ▶ Construction of LCD codes obtained from orbit matrix of p -WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w , using suitable extensions of orbit matrix.
- ▶ Construction of LCD codes obtained from submatrices of orbit matrix of p -WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α , and with f_2 fixed blocks and m block orbits of length p^α , $1 \leq \alpha \leq n$.

LCD codes from p -WSO designs

Let a and d be elements of finite field \mathbb{F}_q , where $q = p^l$ is prime power. Then

$$\det \begin{bmatrix} a & d & \cdots & d \\ d & a & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & a \end{bmatrix}_{n \times n} = (a - d)^{n-1} [a + (n - 1)d].$$

Let M be $b \times v$ incidence matrix of 1 -(v, k, λ) design \mathcal{D} which has b blocks B_1, \dots, B_b .
Let $B_{i,j} = |B_i \cap B_j|$, for all $i, j \in \{1, \dots, b\}$. It follows that

$$[M, x \cdot \mathbf{1}_b, y \cdot \mathbf{1}] \cdot [M, x \cdot \mathbf{1}_b, y \cdot \mathbf{1}]^T = \begin{bmatrix} B_{1,1} + x^2 + y^2 & B_{1,2} + y^2 & \cdots & B_{1,b} + y^2 \\ B_{2,1} + y^2 & B_{2,2} + x^2 + y^2 & \cdots & B_{2,b} + y^2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{b,1} + y^2 & B_{b,2} + y^2 & \cdots & B_{b,b} + x^2 + y^2 \end{bmatrix}$$



LCD codes from p -WSO designs

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

1. If $a = d$ then
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$ and $x^2 + ba \neq 0$, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x \neq 0$ and $b \cdot y^2 + x^2 + b \cdot d \neq 0$
 generate an LCD code over the field \mathbb{F}_q .
2. If $a \neq d$ then
 - the matrix \mathbf{M} for $a + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank,
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 - d + a \neq 0$ and $x^2 + a + (b-1) \cdot d \neq 0$,
 - the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + a + (b-1) \cdot d \neq 0$ and if \mathbf{M} is of full rank, and
 - the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d + a \neq 0$ and $b \cdot y^2 + x^2 + a + (b-1) \cdot d \neq 0$
 generate an LCD code over \mathbb{F}_q .

LCD from orbit matrices of p -WSO designs

Let \mathcal{D} be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design \mathcal{D} .

Let G be an automorphism group of \mathcal{D} which is acting on \mathcal{D} with n point orbits of size w and m block orbits of size w . Let O be an $m \times n$ orbit matrix under the action of G .

- If $a = d$, then

the matrix $[O, x \cdot I_m]$ for $x^2 + mw \cdot d \neq 0$, and

the matrix $[O, x \cdot I_m, y \cdot 1]$ for $x^2 + m \cdot y^2 + mw \cdot d \neq 0$

generate an LCD code over the field \mathbb{F}_q .

- If $a \neq d$, then

the matrix O if O is of full rank and for $a - d \neq 0$,

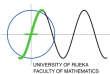
the matrix $[O, x \cdot I_m]$ for $d - q \neq 0$ and $d - a - mw \cdot d \neq 0$,

the matrix $[O, y \cdot 1]$ for $d - a - mw \cdot d - m \cdot y^2 \neq 0$ and if O is of full rank, and

the matrix $[O, x \cdot I_m, y \cdot 1]$ for $d - a - x^2 \neq 0$ and

$$d - a - x^2 - mw \cdot d - m \cdot y^2 \neq 0$$

generate an LCD code over the field \mathbb{F}_q .



LCD codes obtained using submatrices of orbit matrix of p -WSO design (OM1)

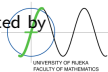
Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q . Let $\mathcal{D} 1-(v, k, r)$ be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} . Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α . Let x and y be non-zero elements of \mathbb{F}_q .

1. For $a = d$, we conclude the following.

- ▶ For $x^2 + f_1 \cdot a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
- ▶ For $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.

2. For $a \neq d$, we conclude the following.

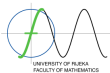
- ▶ If OM1 is of full rank, linear code over the field \mathbb{F}_q generated by matrix $\mathbf{OM1}$ is LCD code.
- ▶ For $x^2 + a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
- ▶ If OM1 is of full rank and for $a + f_1 \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, y \cdot \mathbf{1}]$ is LCD code.
- ▶ For $x^2 + a \neq 0$ and $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.



LCD codes obtained using submatrices of orbit matrix of p -WSO design (OM2)

Let $q = p^l$ and let \mathbb{F}_q be the finite field of order q . Let \mathcal{D} 1- (v, k, r) be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design \mathcal{D} . Let G be an automorphism group of the design \mathcal{D} which acts on the point set of \mathcal{D} with f_1 fixed points and n orbits of length p^α , $1 \leq \alpha \leq l$, and which acts on the block set of the design \mathcal{D} with f_2 fixed blocks and m orbits of length p^α . Let x and y be non-zero elements of \mathbb{F}_q .

1. For $a = d$, we conclude the following.
 - ▶ Linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
 - ▶ For $x^2 + m \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.
2. For $a \neq d$, we conclude the following.
 - ▶ If $\mathbf{OM2}$ is of full rank, linear code over the field \mathbb{F}_q generated by matrix $\mathbf{OM2}$ is LCD code.
 - ▶ For $x^2 + a - d \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
 - ▶ If $\mathbf{OM2}$ is of full rank and for $a - d + m \cdot y^2 \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM2}, y \cdot \mathbf{1}]$ is LCD code.
 - ▶ For $x^2 + a - d \neq 0$ and $x^2 + m \cdot y^2 + a - d \neq 0$, linear code over the field \mathbb{F}_q generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.



LCD codes from A_5

We constructed all weakly self-orthogonal 1-designs and corresponding binary LCD codes constructed from transitive permutation group $G \cong A_5$ and $P < G$, $P \neq I$.

We obtained 4 optimal LCD codes and 2 near-optimal LCD codes.

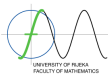
Design	C	$\text{Aut}(C)$
1-(20, 12, 3)	$[25, 5, 11]^*$	$E_{24} : (E_{24} : S_5)$
1-(20, 14, 7)	$[30, 10, 9]^* *$ $[31, 10, 10]^*$	S_5
1-(20, 15, 9)	$[20, 12, 4]^*$ $[21, 12, 4]^* *$	$Z_2 \times S_5$
1-(12, 10, 5)	$[18, 6, 6]^* *$	$(E_{25} : A_6) : E_{22}$
1-(10, 6, 3)	$[11, 5, 4]^*$	S_5
1-(10, 5, 3)	$[10, 6, 3]^*$ $[11, 6, 4]^*$	S_5

Some examples...

We constructed examples of weakly 3-self-orthogonal designs and weakly 5-self-orthogonal designs from permutation representation of the group $S_4(9)$ on 1640 points.

The orbit matrices of the constructed designs were obtained under the action of the cyclic group of orders 3 and 5, which acts on the points of the design without fixed points (all point and block orbits are of the same length).

Design	C
1-(1640, 729, 729)	$\begin{bmatrix} 657, 328, 2 \end{bmatrix}_3$
1-(1640, 1458, 729)	$\begin{bmatrix} 492, 164, 2 \end{bmatrix}_3$
1-(1640, 1638, 819)	$\begin{bmatrix} 493, 164, 2 \end{bmatrix}_3$
1-(1640, 2, 1)	$\begin{bmatrix} 329, 164, 1 \end{bmatrix}_3$
1-(1640, 182, 91)	$\begin{bmatrix} 328, 164, 2 \end{bmatrix}_3$
1-(1640, 911, 911)	$\begin{bmatrix} 329, 164, 3 \end{bmatrix}_3$
1-(1640, 1458, 729)	$\begin{bmatrix} 493, 164, 4 \end{bmatrix}_3$
1-(1640, 1638, 819)	$\begin{bmatrix} 656, 328 \end{bmatrix}_3$
1-(1640, 1458, 729)	$\begin{bmatrix} 328, 164, 2 \end{bmatrix}_5$
1-(1640, 1458, 729)	$\begin{bmatrix} 492, 164, 12 \end{bmatrix}_5$
1-(1640, 1458, 729)	$\begin{bmatrix} 493, 164, 12 \end{bmatrix}_5$
1-(1640, 1638, 819)	$\begin{bmatrix} 329, 164, 3 \end{bmatrix}_5$
1-(1640, 1638, 819)	$\begin{bmatrix} 493, 164, 3 \end{bmatrix}_5$
1-(1640, 1638, 819)	$\begin{bmatrix} 493, 164, 4 \end{bmatrix}_5$



Some examples...

Using weakly 3-self-orthogonal 1-designs constructed from the group $S_4(9)$ on 1640 points, we have constructed LCD codes using Theorem 3. The orbit matrices of the designs are obtained under the action of a cyclic group of order 3 acting on the points of the designs of lengths 1 and 3.

Design	C
1-(1640, 182, 91)	$[57, 19, 1]_3$
	$[58, 19, 2]_3$
	$[1068, 534]_3$
	$[1069, 534]_3$
-----	-----
1-(1640, 729, 729)	$[76, 38, 1]_3$
	$[801, 267]_3$
	$[802, 267]_3$
1-(1640, 1638, 819)	$[39, 19, 1]_3$
	$[534, 267, 2]_3$
	$[535, 267, 3]_3$
1-(1640, 2, 1)	$[38, 19, 2]_3$
	$[534, 267, 2]_3$
	$[535, 267, 3]_3$
1-(1640, 182, 91)	$[58, 19, 2]_3$
	$[801, 267]_3$
	$[802, 267]_3$
-----	-----
1-(1640, 911, 911)	$[76, 38, 2]_3$
	$[77, 38, 2]_3$
	$[1068, 534]_3$
	$[1069, 534]_3$

Designs and graphs

- ▶ The incidence matrix of a symmetric 1-design is the adjacency matrix of a regular digraph.
- ▶ The incidence matrix of a symmetric 1-design with symmetric incidence matrix is the adjacency matrix of a regular graph.

Quasi-strongly regular digraphs

A **quasi-strongly regular digraph**¹ (QSRD) \mathcal{G} with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ is a k -regular digraph on n vertices such that

- ▶ each vertex is incident with t undirected edges,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is a if $x \rightarrow y$,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is c_i , for $i \in \{1, \dots, p\}$, if $x \nrightarrow y$
- ▶ for each c_i , $i \in \{1, \dots, p\}$, there exist two distinct vertices $x \nrightarrow y$ such that the number of paths of length 2 from x to y is c_i .

Number p is **grade** of \mathcal{G} and $c_1 > c_2 > \dots > c_p$.

- ▶ If $p = 1$, a quasi-strongly regular digraph is **strongly regular digraph** (SRD) with parameters (n, k, a, c_1, t) .
- ▶ If $p = 1$ and $k = t$, a quasi-strongly regular digraph is **strongly regular graph**.

¹D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs*, *Graphs and Combinatorics*, **38** (2022)

If A is adjacency matrix of a QSRD with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ then:

- ▶ A is square matrix and $AJ_n = J_nA = kJ$
- ▶ $A^2 = tI_n + aA + c_1C_1 + c_2C_2 + \dots + c_pC_p$ for some non-zero $(0, 1)$ -matrices C_1, C_2, \dots, C_p such that $C_1 + C_2 + \dots + C_p = J_n - I_n - A$
- ▶ $(A^T)^2 = tI_n + aA^T + c_1C_1^T + c_2C_2^T + \dots + c_pC_p^T$ for some non-zero $(0, 1)$ -matrices C_1, C_2, \dots, C_p^T such that $C_1^T + C_2^T + \dots + C_p^T = J_n - I_n - A^T$

From that one can eliminate some graphs whose adjacency matrix will generate non-interesting SO or LCD codes.

Some examples...

Binary SO codes constructed from SRDs and QSRDs on 12 vertices

SRD	code
(12,4,0,2,2)	[12,3,4]
(12,5,2,2,3)	[12,3,6]*
(12,6,5,2,2,3)	[12,4,4]

QSRD or its complement	code
(12,2,0,0;1,0)	[12,6,2]
(12,4,0,0;4,0)	[12,3,4]
(12,4,3,0;3,2,0)	[12,4,4]
(12,4,3,0;3,1)	[12,5,4]*
(12,1,0,0;1,0)	[24,12,2]
(12,3,0,0;3,2,0)	[24,12,4]
(12,5,4,0;5,4,0)	[24,12,4]
(12,3,2,1;1,0)	[12,4,4]
(12,3,2,1;2,0)	[12,5,4]*
(12,5,1,2;4,2)	[12,6,4]*
(12,5,3,2;4,1)	[12,6,2]
(12,4,0,1;4,2,1)	[24,12,8]*

Thank you!

Σας ευχαριστώ!