Self-orthogonal and LCD codes related to some combinatorial structures

Ivona Traunkar
(inovak@math.uniri.hr)
joint work with Vedrana Mikulić Crnković

Faculty of Mathematics, University of Rijeka

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Preliminaries

Self-orthogonal codes from p-WSO designs

LCD codes from p-WSO designs

Self-orthogonal codes from digraphs

Codes

We will talk only about linear codes, i.e. subspaces of the ambient vector space over a field $\mathbb{F}_q$ of order $q = p^l$, where p is prime.
A code C of length n and dimension k is denoted by $[n, k]$.
By $[n, k, d]_q$, we denote code C with minimal distance d over the field $\mathbb{F}_q$.
Specially, if $q = 2$, parameters of code C are denoted by $[n, k, d]$.

A generator matrix of a $[n, k]$ code C is a $k \times n$ matrix whose rows generate all the words of C.

The dual code of a code C is code $C^\perp$, $C^\perp = \{v \in (\mathbb{F}_q)^n \mid (v, c) = 0, \ \forall c \in C\}$.

A code is self-orthogonal if $C \subseteq C^\perp$. A code is self-dual if $C = C^\perp$.
A code is LCD (Linear code with complementary dual) if $C \cap C^\perp = \{\mathbf{0}\}$.

A code C with generator matrix G is LCD code if and only if $\det(G \cdot G^T) \neq 0$.

Weakly self-orthogonal designs

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is called a $t-(v,k,\lambda)$ design, if $\mathcal{P}$ contains v points, every block $B \in \mathcal{B}$ is incident with k points, and every t distinct points are incident with λ blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A design is weakly p-self-orthogonal (p-WSO) if all block intersection numbers gives the same residue modulo p.
A weakly p-self-orthogonal design is p-self-orthogonal if block intersection numbers and the block sizes are multiples of p.

Specially, weakly 2-self-orthogonal design is called weakly self-orthogonal (WSO) design, and 2-self-orthogonal design is called self-orthogonal.

Orbit matrices of a design

Let $\mathcal{D}$ be a 1-(v,k,λ) design and G be an automorphism group of the design. Let $v_1 = |\mathcal{V}_1|, \dots, v_n = |\mathcal{V}_n|$ be the sizes of point orbits and $b_1 = |\mathcal{B}_1|, \dots, b_m = |\mathcal{B}_m|$ be the sizes of block orbits under the action of the group G. We define an orbit matrix as $m \times n$ matrix

$$O = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij} is the number of points of the orbit $\mathcal{V}_j$ incident with a block of the orbit $\mathcal{B}_i$. It is easy to see that the matrix is well-defined and that $k = \sum_{j=1}^{n} a_{ij}$.

For $x \in \mathcal{B}_s$, by counting the incidence pairs (P, x') such that $x' \in \mathcal{B}_t$ and P is incident with the block x, we obtain

$$\sum_{x' \in \mathcal{B}_t} |x \cap x'| = \sum_{j=1}^{m} \frac{b_t}{v_j} a_{sj} a_{tj} = \frac{b_t}{v_j} O[s] \cdot O[t],$$

where $O[s]$ is the s-th row of the matrix O.

Let $\mathcal{D}$ be a weakly p-self-orthogonal design such that

$$k \equiv a \pmod{p}$$

and

$$|B_i \cap B_j| \equiv d \pmod{p},$$

for all $i, j \in \{1, \dots, b\},\ i \neq j$, where B_i and B_j are two blocks of a design $\mathcal{D}$.

Let G be an automorphism group of the design which acts on $\mathcal{D}$ with n point orbits of length w and block orbits of length $b_1, b_2, \dots, b_m$, and let O be an orbit matrix of a design $\mathcal{D}$ under the action of a group G.

Let $q = p^n$ be prime power and let $\mathbb{F}_q$ be a finite field of order q.
In $\mathbb{F}_q$, for $x \in \mathcal{B}_s,\ s \neq t$ it follows that

$$\frac{b_t}{w} O[s] \cdot O[t] = b_t d, \qquad (1)$$

$$\frac{b_s}{w} O[s] \cdot O[s] = a + (b_s - 1)d. \qquad (2)$$

Submatrices of orbit matrix of p-WSO design

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q.
Let $\mathcal{D}$ 1-(v, k, r) be a design sunch that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\},\ i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$.

Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length $p^\alpha,\ 1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α.

$$O = \left[\begin{array}{cccc|cccc}
 & & & & a_{1,f_1+1} & a_{1,f_1+2} & \cdots & a_{1,f_1+n} \\
 & \text{OM1} & & & a_{2,f_1+1} & a_{2,f_1+2} & \cdots & a_{2,f_1+n} \\
 & & & & \vdots & \vdots & \ddots & \vdots \\
 & & & & a_{f_2,f_1+1} & a_{f_2,f_1+2} & \cdots & a_{f_2,f_1+n} \\
\hline
a_{f_2+1,1} & a_{f_2+1,2} & \cdots & a_{f_2+1,f_1} & & & & \\
a_{f_2+2,1} & a_{f_2+1,2} & \cdots & a_{f_2+2,f_1} & & & & \\
\vdots & \vdots & \ddots & \vdots & & \text{OM2} & & \\
a_{f_2+m,1} & a_{f_2+m,2} & \cdots & a_{f_2+m,f_1} & & & &
\end{array}\right]$$

Self-orthogonal codes from p-WSO designs

Let $\mathbb{F}_q$ be a finite field of order $q = p^l$, where p is a prime.

- Construction of SO codes obtained from incidence matrix of p-WSO designs, using suitable extensions of incidence matrix of a design.
- Construction od SO codes obtained from orbit matrix of p-WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w, using suitable extensions of orbit matrix.
- Construction of SO codes obtained from submatrices of orbit matrix of of p-WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α, and with f_2 fixed blocks and m block orbits of length p^α, $1 \leqslant \alpha \leqslant n$.

SO codes from incidence matrix of WSO 1-designs

Theorem (V. Tonchev)

Let $\mathcal{D}$ be weakly self-orthogonal design and let M be it's $b \times v$ incidence matrix.

1. *If $\mathcal{D}$ is a self-orthogonal design, then the matrix M generates a binary self-orthogonal code.*
2. *If $\mathcal{D}$ is such that k is even and the block intersection numbers are odd, then the matrix $[I_b, M, \mathbf{1}]$ generates a binary self-orthogonal code.*
3. *If $\mathcal{D}$ is such that k is odd and the block intersection numbers are even, then the matrix $[I_b, M]$ generates a binary self-orthogonal code.*
4. *If $\mathcal{D}$ is such that k is odd and the block intersection numbers are odd, then the matrix $[M, \mathbf{1}]$ generates a binary self-orthogonal code.*

SO codes from incidence matrix of p-WSO 1-designs

Let $q = p^l$ be prime power and $\mathbb{F}_q$ a finite field of order q. Let $\mathcal{D}$ be a weakly p-self-orthogonal design such that $k \equiv a \pmod p$ and $|B_i \cap B_j| \equiv d \pmod p$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design $\mathcal{D}$. Let M be it's $b \times v$ incidence matrix.

1. If $\mathcal{D}$ is p-self-orthogonal design, then M generates a self-orthogonal code over $\mathbb{F}_q$.
2. If $a = 0$ and $d \neq 0$, then the matrix $[\sqrt{d} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in $\mathbb{F}_q$, and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
3. If $a \neq 0$ and $d = 0$, then the matrix $[M, \sqrt{-a} \cdot I_b]$ generates a self-orthogonal code over $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in $\mathbb{F}_q$, and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
4. If $a \neq 0$ and $d \neq 0$, there are two cases:
   - if $a = d$, then the matrix $[M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in $\mathbb{F}_q$, and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise, and
   - if $a \neq d$, then the matrix $[\sqrt{d-a} \cdot I_b, M, \sqrt{-d} \cdot \mathbf{1}]$ generates a self-orthogonal code over $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-d$ is a square in $\mathbb{F}_q$, and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.

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SO codes form orbit matrices of p-WSO 1-designs

Let $q = p^l$ be prime power and $\mathbb{F}_q$ a finite field of order p.
Let $\mathcal{D}$ be a weakly p-self-orthogonal 1-design such that $k \equiv a \pmod p$ and $|B_i \cap B_j| \equiv d \pmod p$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are two blocks of a design $\mathcal{D}$, and let G be an automorphism group of the design which acts on $\mathcal{D}$ with n point orbits of length w and m block orbits of length w and let O be the orbit matrix of $\mathcal{D}$ under action of a group G.

- If $a = d$ we differ two cases.
  - a) If $p \mid w$, the linear code spanned by the rows of the matrix O is self-orthogonal code over the field $\mathbb{F}_q$.
  - b) If $p \nmid w$, the linear code spanned by the rows of the matrix $[O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-wd$ is a square in the field $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
- If $a \neq d$, we differ three cases.
  - a) If $p \mid w$, the linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, O]$ is self-orthogonal code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $d - a$ kvadrat u polju $\mathbb{F}_q$ i $\mathbb{F} = \mathbb{F}_{q^2}$ inače.
  - b) If $p \mid w - 1$, the linear code spanned by the rows of the matrix $[\sqrt{wd-a} \cdot I_m, O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal $[m+n, m]$ code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $wd - a$ and $-wd$ are squares in the field $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise. If $m = n$, the obtained code is self-dual.
  - c) If $p \nmid w$ i $q \nmid w - 1$, the linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, O, \sqrt{-wd} \cdot \mathbf{1}]$ is self-orthogonal $[m+n+1, m]$ code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $d - a$ and $-wd$ are squares in the filed $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.

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SO codes obtained using submatrices of orbit matrix of p-WSO design

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q.
Let $\mathcal{D}$ 1-(v, k, r) be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$. Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^α, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α.

- If $a = d$, we differ two cases.
  1. Linear code spanned by the rows of the matrix $[OM1, \sqrt{-a} \cdot \mathbf{1}]$ is self-orthogonal code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $-a$ is a square in the field $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
  2. Linear code spanned by the rows of the matrix $OM2$ is self-orthogonal code over the field $\mathbb{F}_q$.
- If $a \neq d$, we differ two cases.
  1. Linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_{f_2}, OM1, \sqrt{-d} \cdot \mathbf{1}]$ is self-orthogonal $[f_2 + f_1 + 1, f_2]$ code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $d - a$ and $-d$ are squares in the field $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise.
  2. Linear code spanned by the rows of the matrix $[\sqrt{d-a} \cdot I_m, OM2]$ is self-orthogonal $[m + n, m]$ code over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{F}_q$ if $d - a$ is a square in the filed $\mathbb{F}_q$ and $\mathbb{F} = \mathbb{F}_{q^2}$ otherwise. If $m = n$, the obtained code is self-dual.

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Some results...

Code	Design
$[6, 2, 4]$*	1-$(22, 10, 10)$
$[10, 4, 4]$*	1-$(110, 72, 36)$
$[12, 2, 8]$*	1-$(132, 100, 100)$
$[12, 3, 6]$*	1-$(132, 66, 6)$
$[12, 5, 4]$*	1-$(132, 40, 40)$
$[12, 5, 4]$*	1-$(132, 46, 46)$
$[12, 6, 4]$*	1-$(66, 45, 45)$
$[12, 11, 2]$*	1-$(132, 22, 2)$
$[15, 4, 8]$*	1-$(165, 48, 48)$
$[24, 12, 8]$*	1-$(132, 27, 27)$
$[16, 5, 8]$*	1-$(165, 109, 109)$
$[31, 15, 8]$*	1-$(165, 116, 116)$
$[9, 2, 6]_3$*	1-$(30, 21, 7)$
$[6, 3, 2]$**	1-$(22, 2, 1)$
$[10, 3, 4]$**	1-$(66, 46, 46)$
$[14, 6, 4]$**	1-$(110, 72, 72)$
$[64, 4, 32]$**	1-$(132, 55, 5)$
$[30, 4, 18]_3$**	1-$(30, 18, 3)$
$[96, 48, 16]+$	1-$(144, 23, 23)$

Table: Optimal($*$), near-optimal($**$) and best known($+$) SO codes

LCD codes from p-WSO designs

Let $\mathbb{F}_q$ be a finite field of order $q = p^l$, where p is a prime.

- Construction of LCD codes obtained from incidence matrix of p-WSO designs, using suitable extensions of incidence matrix of a design.
- Construction od LCD codes obtained from orbit matrix of p-WSO designs under the action of group G which acts on design with n point orbits of length w and m block orbits of length w, using suitable extensions of orbit matrix.
- Construction of LCD codes obtained from submatrices of orbit matrix of of p-WSO designs under the action of group G which acts on design with f_1 fixed points and n point orbits of length p^α, and with f_2 fixed blocks and m block orbits of length p^α, $1 \leqslant \alpha \leqslant n$.

LCD codes from p-WSO designs

Let a and d be elements of finite filed $\mathbb{F}_q$, where $q = p^l$ is prime power. Then

$$\det \begin{bmatrix} a & d & \cdots & d \\ d & a & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & a \end{bmatrix}_{n \times n} = (a-d)^{n-1}[a+(n-1)d].$$

Let M be $b \times v$ incidence matrix of 1-(v,k,λ) design $\mathcal{D}$ which has b blocks $B_1, \ldots, B_b$. Let $B_{i,j} = |B_i \cap B_j|$, for all $i, j \in \{1, \ldots, b\}$. It follows that

$$[M, x \cdot I_b, y \cdot \mathbf{1}] \cdot [M, x \cdot I_b, y \cdot \mathbf{1}]^T = \begin{bmatrix} B_{1,1} + x^2 + y^2 & B_{1,2} + y^2 & \cdots & B_{1,b} + y^2 \\ B_{2,1} + y^2 & B_{2,2} + x^2 + y^2 & \cdots & B_{2,b} + y^2 \\ \vdots & \vdots & \ddots & \vdots \\ B_{b,1} + y^2 & B_{b,2} + y^2 & \cdots & B_{b,b} + x^2 + y^2 \end{bmatrix}$$

LCD codes from p-WSO designs

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

1. If $a = d$ then
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x \neq 0$ and $x^2 + ba \neq 0$, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x \neq 0$ and $b \cdot y^2 + x^2 + b \cdot d \neq 0$
   generate an LCD code over the field $\mathbb{F}_q$.
2. If $a \neq d$ then
   the matrix $\mathbf{M}$ for $a + (b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank,
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b]$ for $x^2 - d + a \neq 0$ and $x^2 + a + (b-1) \cdot d \neq 0$,
   the matrix $[\mathbf{M}, y\mathbf{1}]$ for $by^2 + a + (b-1) \cdot d \neq 0$ and if $\mathbf{M}$ is of full rank, and
   the matrix $[\mathbf{M}, x \cdot \mathbf{I}_b, y\mathbf{1}]$ for $x^2 - d + a \neq 0$ and $b \cdot y^2 + x^2 + a + (b-1) \cdot d \neq 0$
   generate an LCD code over $\mathbb{F}_q$.

LCD from orbit matrices of p-WSO designs

Let $\mathcal{D}$ be such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\},\ i \neq j$, where B_i and B_j are two blocks of the design $\mathcal{D}$.

Let G be an automorphism group of $\mathcal{D}$ which is acting od $\mathcal{D}$ with n point orbits of size w and m block orbits of size w. Let O be an $m \times n$ orbit matrix under the action of G.

- If $a = d$, then
  the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ for $x^2 + mw \cdot d \neq 0$, and
  the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $x^2 + m \cdot y^2 + mw \cdot d \neq 0$
  generate an LCD code over the field $\mathbb{F}_q$.
- If $a \neq d$, then
  the matrix $\mathbf{O}$ if O is of full rank and for $a - d \neq 0$,
  the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m]$ for $d - q \neq 0$ and $d - a - mw \cdot d \neq 0$,
  the matrix $[\mathbf{O}, y \cdot \mathbf{1}]$ for $d - a - mw \cdot d - m \cdot y^2 \neq 0$ and if O is of full rank, and
  the matrix $[\mathbf{O}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ for $d - a - x^2 \neq 0$ and
  $d - a - x^2 - mw \cdot d - m \cdot y^2 \neq 0$
  generate an LCD code over the field $\mathbb{F}_q$.

LCD codes obtained using submatrices of orbit matrix of p-WSO design ($OM1$)

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q. Let $\mathcal{D}$ 1-(v, k, r) be a design sunch that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \ldots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$. Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^α, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α. Let x and y be non-zero elements of $\mathbb{F}_q$.

1. For $a = d$, we conclude the following.
   - For $x^2 + f_1 \cdot a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
   - For $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.
2. For $a \neq d$, we conclude the following.
   - If $OM1$ is of full rank, linear code over the filed $\mathbb{F}_q$ generated by matrix $\mathbf{OM1}$ is LCD code.
   - For $x^2 + a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}]$ is LCD code.
   - If $OM1$ is of full rank and for $a + f_1 \cdot y^2 \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, y \cdot \mathbf{1}]$ is LCD code.
   - For $x^2 + a \neq 0$ and $x^2 + f_1 \cdot y^2 + f_1 \cdot a \neq 0$, linear code over the filed $\mathbb{F}_q$ generated by matrix $[\mathbf{OM1}, x \cdot \mathbf{I}_{f_1}, y \cdot \mathbf{1}]$ is LCD code.

LCD codes obtained using submatrices of orbit matrix of p-WSO design ($OM2$)

Let $q = p^l$ and let $\mathbb{F}_q$ be the finite field of order q. Let $\mathcal{D}$ 1-(v, k, r) be a design such that $k \equiv a \pmod{p}$ and $|B_i \cap B_j| \equiv d \pmod{p}$, for all $i, j \in \{1, \dots, b\}$, $i \neq j$, where B_i and B_j are blocks of the design $\mathcal{D}$. Let G be an automorphism group of the design $\mathcal{D}$ which acts on the point set of $\mathcal{D}$ with f_1 fixed points and n orbits pf length p^α, $1 \leqslant \alpha \leqslant l$, and which acts of block set of the design $\mathcal{D}$ with f_2 fixed blocks and m orbits of length p^α. Let x and y be non-zero elements of $\mathbb{F}_q$.

1. For $a = d$, we conclude the following.
   - Linear code over the fileld $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
   - For $x^2 + m \cdot y^2 \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.
2. For $a \neq d$, we conclude the following.
   - If $OM2$ is of full rank, linear code over the field $\mathbb{F}_q$ generated by matrix $\mathbf{OM2}$ is LCD code.
   - For $x^2 + a - d \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m]$ is LCD code.
   - If $OM2$ is of full rank and for $a - d + m \cdot y^2 \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, y \cdot \mathbf{1}]$ is LCD code.
   - For $x^2 + a - d \neq 0$ and $x^2 + m \cdot y^2 + a - d \neq 0$, linear code over the field $\mathbb{F}_q$ generated by matrix $[\mathbf{OM2}, x \cdot \mathbf{I}_m, y \cdot \mathbf{1}]$ is LCD code.

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LCD codes from A_5

We constructed all weakly self-orthogonal 1-designs and corresponding binary LCD codes constructed from transitive permutation group $G \cong A_5$ and $P < G,\ P \neq I$.

We obtained 4 optimal LCD codes and 2 near-optimal LCD codes.

Design	C	**Aut**(C)
1-(20, 12, 3)	$[25, 5, 11]*$	$E_{2^4} : (E_{2^4} : S_5)$
1-(20, 14, 7)	$[30, 10, 9] **$ $[31, 10, 10]*$	S_5
1-(20, 15,9)	$[20, 12, 4]*$ $[21, 12, 4] **$	$Z_2 \times S_5$
1-(12,10,5)	$[18, 6, 6] **$	$(E_{2^5} : A_6) : E_{2^2}$
1-(10,6,3)	$[11, 5, 4]*$	S_5
1-(10,5,3)	$[10, 6, 3]*$ $[11,6,4]*$	S_5

Some examples...

We constructed examples of weakly 3-self-orthogonal designs and weakly 5-self-orthogonal designs from permutation representation of the group $S_4(9)$ on 1640 points.
The orbit matrices of the constructed designs were obtained under the action of the cyclic group of orders 3 and 5, which acts on the points of the design without fixed points (all point and block orbits are of the same length).

Design	C
1-$(1640, 729, 729)$	$[657, 328, 2]_3$
1-$(1640, 1458, 729)$	$[492, 164, 2]_3$ $[493, 164, 2]_3$
1-$(1640, 1638, 819)$	$[329, 164, 1]_3$
1-$(1640, 2, 1)$	$[328, 164, 2]_3$ $[329, 164, 3]_3$
1-$(1640, 182, 91)$ 1-$(1640, 911, 911)$	$[493, 164, 4]_3$ $[656, 328]_3$
1-$(1640, 1458, 729)$	$[328, 164, 2]_5$ $[492, 164, 12]_5$ $[493, 164, 12]_5$ $[329, 164, 3]_5$
1-$(1640, 1638, 819)$	$[493, 164, 3]_5$ $[493, 164, 4]_5$

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Some examples...

Using weakly 3-self-orthogonal 1-designs constructed from the group $S_4(9)$ on 1640 points, we have constructed LCD codes using Theorem 3. The orbit matrices of the designs are obtained under the action of a cyclic group of order 3 acting on the points of the designs of lengths 1 and 3.

Design	C
1-$(1640, 182, 91)$	$[57, 19, 1]_3$ $[58, 19, 2]_3$ $[1068, 534]_3$ $[1069, 534]_3$
1-$(1640, 729, 729)$	$[76, 38, 1]_3$ $[801, 267]_3$ $[802, 267]_3$
1-$(1640, 1638, 819)$	$[39, 19, 1]_3$ $[534, 267, 2]_3$ $[535, 267, 3]_3$
1-$(1640, 2, 1)$	$[38, 19, 2]_3$ $[534, 267, 2]_3$ $[535, 267, 3]_3$
1-$(1640, 182, 91)$	$[58, 19, 2]_3$ $[801, 267]_3$ $[802, 267]_3$
1-$(1640, 911, 911)$	$[76, 38, 2]_3$ $[77, 38, 2]_3$ $[1068, 534]_3$ $[1069, 534]_3$

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Designs and graphs

- The incidence matrix of a symmetric 1-design is the adjacency matrix of a regular digraph.
- The incidence matrix of a symmetric 1-design with symmetric incidence matrix is the adjacency matrix of a regular graph.

Quasi-strongly regular digraphns

A quasi-strongly regular digraph[1] (QSRD) $\mathcal{G}$ with parameters $(n, k, t, a; c_1, c_2, \ldots, c_p)$ is a k-regular digraph on n vertices such that

- each vertex is incident with t undirected edges,
- for any two distinct vertices x, y the number of paths of length 2 from x to y is a if $x \to y$,
- for any two distinct vertices x, y the number of paths of length 2 from x to y is c_i, for $i \in \{1, \ldots, p\}$, if $x \nrightarrow y$
- for each c_i, $i \in \{1, \ldots, p\}$, there exist two distinct vertices $x \nrightarrow y$ such that the number of paths of length 2 from x to y is c_i.

Number p is grade of $\mathcal{G}$ and $c_1 > c_2 > \cdots > c_p$.

- If $p = 1$, a quasi-strongly regular digraph is strongly regular digraph (SRD) with paramters (n, k, a, c_1, t).
- If $p = 1$ and $k = t$, a quasi-strongly regular digraph is strongly regular graph.

[1] D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs, Graphs and Combinatorics*, 38, (2022)

If A is adjacency matrix of a QSRD with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ then:

- A is square matrix and $AJ_n = J_nA = kJ$
- $A^2 = tI_n + aA + c_1C_1 + c_2C_2 + \cdots + c_pC_p$ for some non-zero $(0,1)$-matrices $C_1, C_2, \dots, C_p$ such that $C_1 + C_2 + \cdots + C_p = J_n - I_n - A$
- $(A^T)^2 = tI_n + aA^T + c_1C_1^T + c_2C_2^T + \cdots + c_pC_p$ for some non-zero $(0,1)$-matrices $C_1, C_2, \dots, C_p^T$ such that $C_1^T + C_2^T + \cdots + C_p^T = J_n - I_n - A^T$

From that one can eliminate some graphs whose adjacency matrix will generate non-interesting SO or LCD codes.

Some examples...
Binary SO codes constructed from SRDs and QSRDs on 12 vertices

SRD	code
(12,4,0,2,2)	[12,3,4]
(12,5,2,2,3)	[12,3,6]*
(12,65,2,2,3)	[12,4,4]

QSRD or its complement	code
(12,2,0,0;1,0)	[12,6,2]
(12,4,0,0;4,0)	[12,3,4]
(12,4,3,0;3,2,0)	[12,4,4]
(12,4,3,0;3,1)	[12,5,4]*
(12,1,0,0;1,0)	[24,12,2]
(12,3,0,0;3,2,0)	[24,12,4]
(12,5,4,0;5,4,0)	[24,12,4]
(12,3,2,1;1,0)	[12,4,4]
(12,3,2,1;2,0)	[12,5,4]*
(12,5,1,2;4,2)	[12,6,4]*
(12,5,3,2;4,1)	[12,6,2]
(12,4,0,1;4,2,1)	[24,12,8]*

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Thank you!

Σας ευχαριστώ!