An alternative approach to the Five Line Conjecture

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(joint work with Riste Škrekovski)

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Introduction

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A **cubic graph** G is a graph in which every vertex has degree 3.

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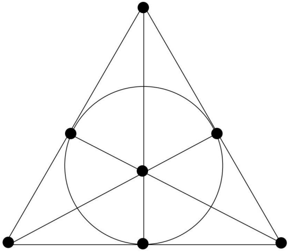

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- 4-line coloring is a theoretical minimum.

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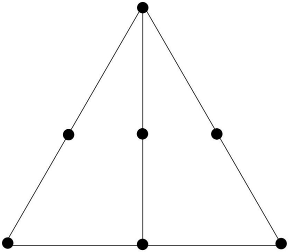

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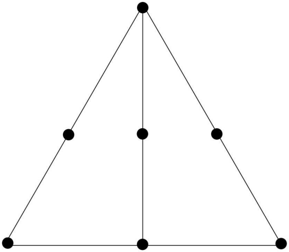

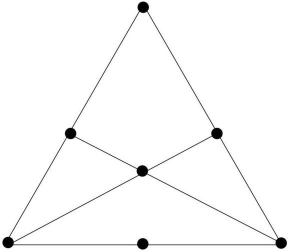

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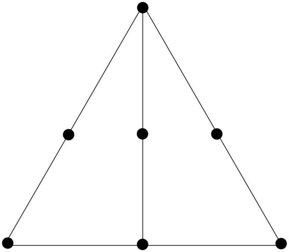

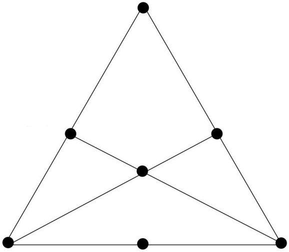

Remark. Five-line conjecture can be restated in terms of proper abelian colorings.

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- the existence of $\mathbb{Z}_4 \times \mathbb{Z}_2$-coloring implies the existence of coloring by all three remaining exceptional groups.

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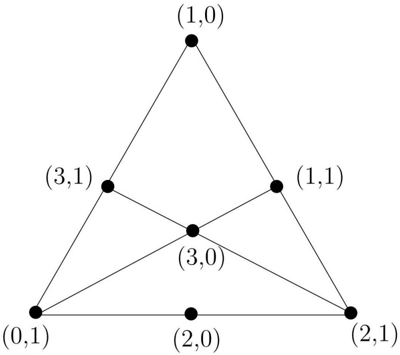

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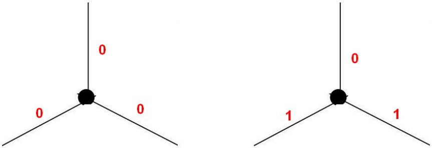

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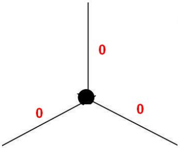

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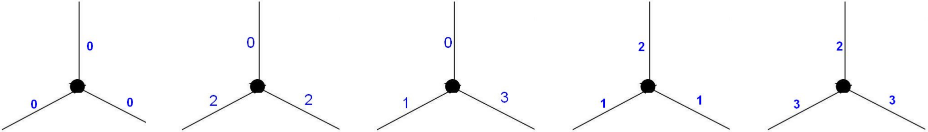

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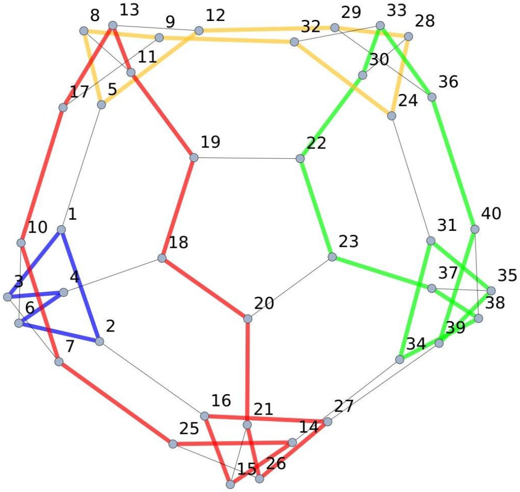

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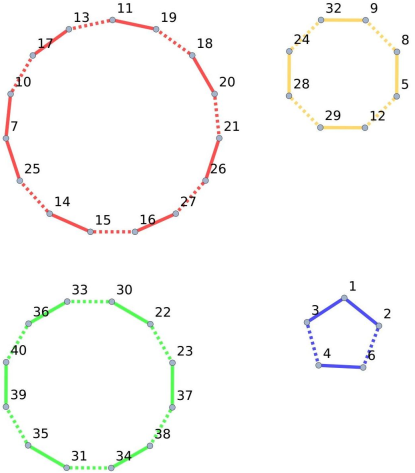

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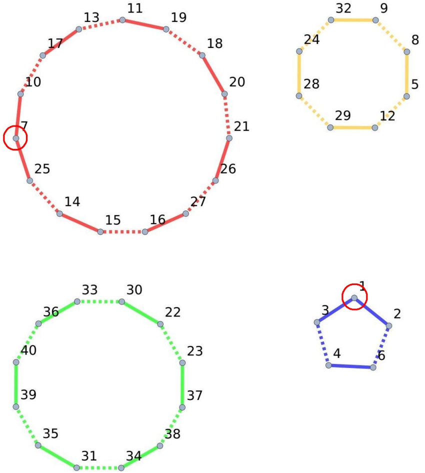

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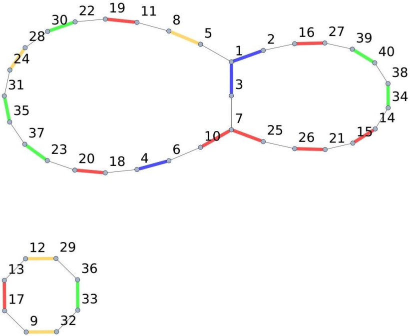

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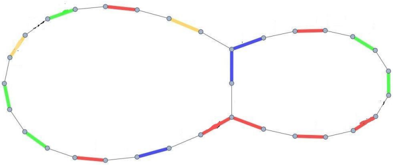

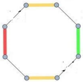

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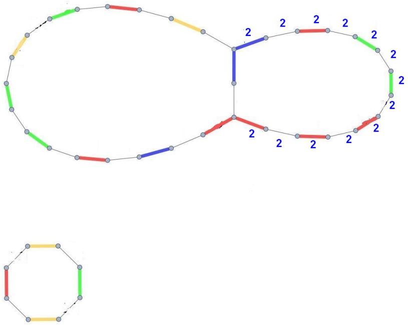

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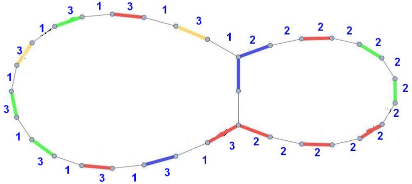

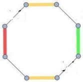

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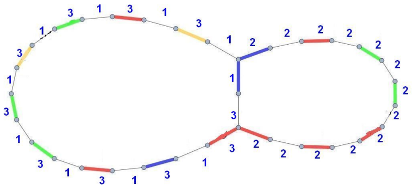

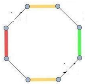

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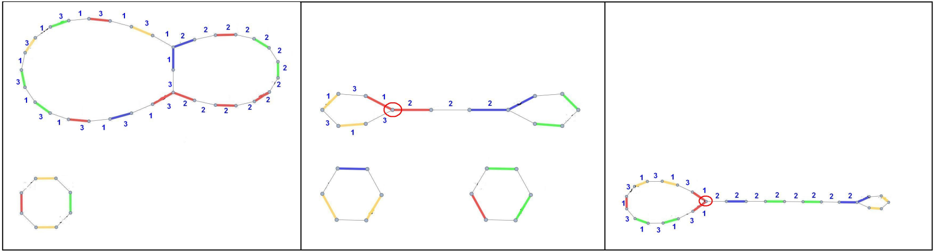

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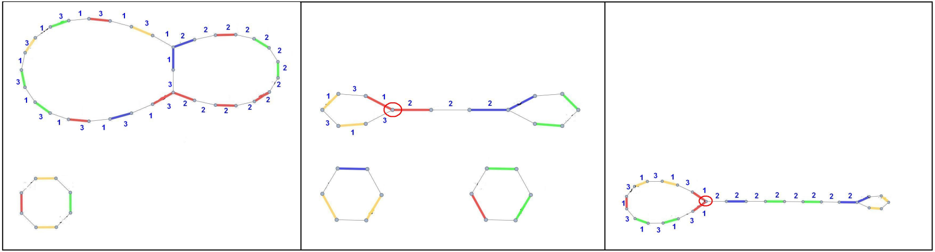

Theorem. A cubic graph G has a proper $\mathbb{Z}_4 \times \mathbb{Z}_2$-coloring if and only if there exists a 2-factor F in G and a matching M in F such that:

- $H = G - M$ has an F-matching
- whose F-complement is 3-even.

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Thank you for the attention!