On the algebraic connectivity of a subfamily of generalized Petersen graphs

John Baptist Gauci

joint work with James Zammit

Department of Mathematics, University of Malta, Msida, Malta

Kalamata, Greece — 2 June 2025

Overview

1. Introduction and definitions
   - Preliminaries
   - Bounds
2. Generalized Petersen graphs
   - Definition
   - Algebraic Connectivity

Introduction

Definition

Let G be a graph with adjacency matrix $\mathbf{A} = \mathbf{A}(G)$ and degree matrix $\mathbf{D} = \mathbf{D}(G)$. The **Laplacian matrix** $\mathbf{L}(G)$, is the matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$.

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Remark

$\mathbf{L}(G)$ is a **positive semi-definite matrix** and thus:

- all eigenvalues of $\mathbf{L}(G)$ are non-negative;
- the eigenvector $\mathbf{j}$ corresponds to the eigenvalue 0;
- 0 is the smallest eigenvalue of $\mathbf{L}(G)$.

Fundamental Result

Theorem (Fiedler, 1973)

Let G be a simple graph on $n \geq 2$ vertices and let

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n$$

be the eigenvalues of $\mathbf{L}(G)$. Then, λ_2 is equal to 0 if and only if G is disconnected .

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Due to this relationship, the eigenvalue λ_2

- serves as a key indicator of whether a graph is connected or not,
- is called the **algebraic connectivity** of G,
- is usually denoted by $\alpha(G)$.

Some Bounds

Theorem (Fiedler, 1973)

Let G be a graph with vertex set $V = V(G)$ of size n and edge set of size m. Then

$$\alpha(G) \leq \left(\frac{n}{n-1}\right) \min_{v \in V} \{d_G(v)\} \leq \frac{2m}{n-1}.$$

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Theorem (Fiedler, 1973)

The maximum eigenvalue of $\mathbf{L}(G)$ for a graph G on n vertices is $b(G) = n - \alpha(\overline{G})$, where $\overline{G}$ denotes the complement of G.

Theorem (Fiedler, 1973)

For a graph G with vertex set $V = V(G)$ of size n,

$$\alpha(G) \geq 2 \min_{v \in V} \{d_G(v)\} - n + 2.$$

Example

Consider the complete graph K_n. By the previous three theorems:

- $\alpha(K_n) \leq \left(\frac{n}{n-1}\right) \min_{v \in V} \{d_{K_n}(v)\} = \left(\frac{n}{n-1}\right)(n-1) = n$.
- $\alpha(K_n) \geq 2 \min_{v \in V} \{d_{K_n}(v)\} - n + 2 = 2(n-1) - n + 2 = n$.
- $b(K_n) = n - \alpha(\overline{K_n}) = n - 0 = n$.

Thus, $\alpha(K_n) = b(K_n)$ and the spectrum of $\mathbf{L}(K_n)$ is given by

$$\begin{bmatrix} 0 & n \\ 1 & n-1 \end{bmatrix}.$$

Simple Results

Theorem (Fiedler, 1973)

Let G be a graph. Then $\alpha(G) = b(G)$ if and only if G is either the complete graph or the null graph.

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Theorem (Fiedler, 1973)

Let G be a graph on n vertices having an independent set of vertices of size k. Then $\alpha(G) \leq n - k$.

Relationship with vertex connectivity

The vertex connectivity of a graph G, denoted by $\kappa(G)$, is the least number of vertices that need to be deleted in order to disconnect G.

Theorem (Fiedler, 1973)

Let G be a graph and let the vertex set $V = V(G)$ be decomposed into two disjoint subsets V_1 and V_2 such that $V = V_1 \cup V_2$. Let G_1 and G_2 be subgraphs of G generated by V_1 and V_2 respectively. Then

$$\alpha(G) \leq \min\{\alpha(G_1) + |V_2|, \alpha(G_2) + |V_1|\}.$$

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A direct consequence is the following:

Theorem (Fiedler, 1973)

Let G be a graph such that $G \neq K_n$. Then $\alpha(G) \leq \kappa(G)$.

Generalized Petersen Graph $GP[n, k]$

Definition

For $n \geq 3$ and $1 \leq k < n$, the **generalized Petersen graph** $GP[n, k]$ is the graph with the vertex–set of cardinality $2n$ given by

$$V(GP[n, k]) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and the edge–set of cardinality $3n$ given by

$$E(GP[n, k]) = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i : 0 \leq i \leq n-1\}.$$

Generalized Petersen Graph $GP[n,k]$

The generalized Petersen graph $GP[n,k]$ is made up of an ***outer cycle*** on the outer vertices; $\ell = \gcd(n,k)$ ***inner cycles*** each of length $t = \frac{n}{\ell}$ on the inner vertices; and ***spokes*** connecting an outer vertex u_i with an inner vertex v_i (for $i = 0, \ldots, n-1$).

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In a generalized Petersen graph $GP[n, k]$,

- if n and k are relatively prime, the inner vertices generate a cycle of length n;
- if n and k are not relatively prime, the inner vertices generate a vertex–disjoint union of cycles of the same length.

Generalized Petersen Graph $GP[n, k]$

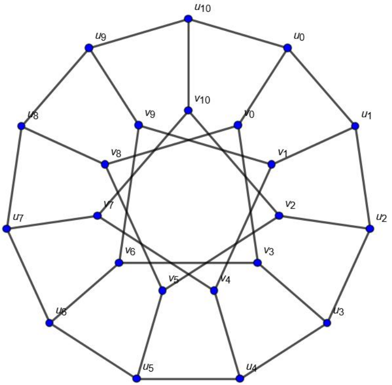

Figure 1: $GP[11, 3]$

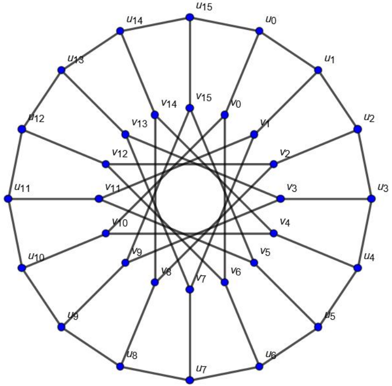

Figure 2: $GP[16, 6]$

$GP[tk, k]$ for integers t and k

The generalized Petersen graph $GP[tk, k]$ for integers $t \geq 3$ and $k \geq 1$ is made up of

- an outer cycle on the vertices u_i;
- k inner cycles each of length t on the vertices v_i; and
- spokes connecting each vertex u_i with the corresponding vertex v_i.

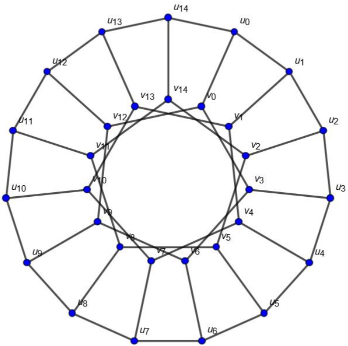

Figure 3: $GP[15, 3]$

Eigenvalues of the Adjacency Matrix of $GP[n, k]$

Note: Since $GP[n, k]$ is cubic, then by knowing the second largest eigenvalue of the adjacency matrix, we would be able to find the algebraic connectivity using $\mathbf{L} = 3\mathbf{I} - \mathbf{A}$.

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Theorem (Gera & Stănică, 2011)

The eigenvalues of $\mathbf{A}(GP[(n,k])$ are given by

$$\cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi jk}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2\pi jk}{n}\right)\right)^2 + 1},$$

for $j \in \{0, \ldots, n-1\}$.

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Corollary

The eigenvalues of $\mathbf{A}(GP[(tk, k])$ are given by $\mu_j^{\pm} =$

$$\cos\left(\frac{2\pi j}{tk}\right) + \cos\left(\frac{2\pi j}{t}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi j}{tk}\right) - \cos\left(\frac{2\pi j}{t}\right)\right)^2 + 1},$$

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Lemma 1

Let $\mu_j^{\pm}$ be the eigenvalues of $\mathbf{A}(GP[(tk,k]))$. Then for fixed k,

$$\mu_t^{+} = \mu_{t(k-1)}^{+} \text{ and } \mu_t^{-} = \mu_{t(k-1)}^{-}$$

for all $t \geq 3$.

Algebraic connectivity of $GP[tk, k]$

For $k \in \{1, 2, 3\}$, we have the following results:

Lemma 2

- $\alpha(GP[t, 1]) = 1$ for $t \in \{3, 4\}$.
- $\alpha(GP[2t, 2]) = 2.23607$ for $t \in \{3, 4, 5\}$.
- $\alpha(GP[3t, 3]) = 2.30278$ for $t \in \{3, 4\}$.

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Conjecture

Let $\mu_j^{\pm}$ be the eigenvalues of $\mathbf{A}(GP[(tk, k)])$ where $k \geq 4$ and $t \geq 3$. For fixed k, if $t \leq k$ then μ_t^{+} is the second largest eigenvalue.

Algebraic connectivity of $GP[tk, k]$

We have partial results for small values of k for the above conjecture.

Method

First note that $\mu_j^+ > \mu_j^-$ for all $j \in \{0, \ldots, tk-1\}$.
Let $x = \cos \frac{2\pi j}{tk}$. We get

$$f^+(x) = x + g(x) + \sqrt{(x - g(x))^2 + 1}$$

where

$$g(x) = \sum_{r=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^r \binom{k}{2r} x^{k-2r}(1-x^2)^r.$$

We show that there is no other x which gives a value of μ_j^+ between μ_t^+ and 3.

Empirical results suggest that the above conjecture is true, however a complete proof is still elusive.

Algebraic connectivity of $GP[tk, k]$

Lemma 1 and an eventual proof of the above conjecture would imply a proof of the following conjecture.

Conjecture 2

Consider the generalized Petersen graph $GP[tk, k]$ for $k \geq 4$ and $t \geq 3$. Then for any value of k, $\alpha(GP[tk, k]) = 3 - \mu_t^+$ for $t \in \{3, \ldots, k\}$, where $\mu_j^\pm$ are the eigenvalues of $\mathbf{A}(GP[(tk, k])$.

Example:

- For $k = 4$ and $t \in \{3, 4\}$, $\alpha(GP[tk, k]) = 0.58579$.
- For $k = 5$ and $t \in \{3, 4, 5\}$, $\alpha(GP[tk, k]) = 0.47548$.
- For $k = 6$ and $t \in \{3, 4, 5, 6\}$, $\alpha(GP[tk, k]) = 0.38197$.
- For $k = 7$ and $t \in \{3, \ldots, 7\}$, $\alpha(GP[tk, k]) = 0.30798$.

Algebraic connectivity of $GP[n, k]$

Theorem (Gera & Stănică, 2011)

Let $\alpha(n, 2)$ denote the algebraic connectivity of $GP[n, 2]$. Then $\lim_{n\to\infty} \alpha(n, 2) = 0$.

Conjecture 3

Let $\alpha(n, k)$ denote the algebraic connectivity of $GP[n, k]$. Then, for $k \geq 4$, $\alpha(n, k)$ is strictly monotonic decreasing for values of $n \geq k^2 + 1$ and $\lim_{n\to\infty} \alpha(n, k) = 0$.

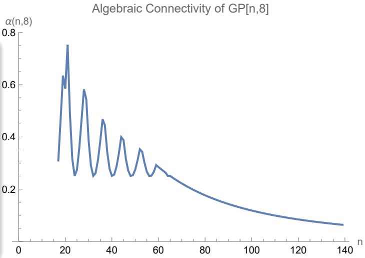

Figure 4: $\alpha(n, 8)$

THANK YOU!