Akademia Górniczo-Hutnicza
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AGH University of Science
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# Further results on decomposition of low degree circulant graphs into cycles

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Joint work with Mariusz Meszka

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# Presentation plan

1. Key definitions: decomposition, circulant, . . .
2. Known results on decomposition of some circulants
3. Main theorem followed by justification in selected cases

## Definition

A *decomposition* of a graph $G$ is a collection $\{H_1, H_2, \ldots, H_t\}$ of edge-disjoint subgraphs of $G$ such that each edge of $G$ belongs to exactly one $H_i$.

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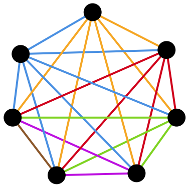
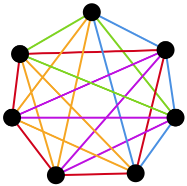


$\{S_1, \ldots, S_6\}$

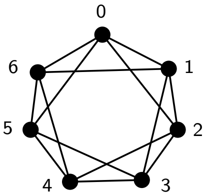


$\{C_3, C_4, C_4, C_5, C_5\}$

Figure: Decomposition of $K_7$ into stars and into cycles

## Definition

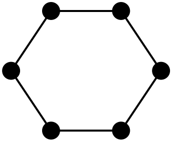
For a positive integer $n$ and a set $S \subseteq \{1, \ldots, \lfloor(\frac{n}{2}\rfloor\}$ a *circulant* is a graph $G = (V, E)$ such that $V = \mathbb{Z}_n$ and $E = \{\{u, v\} : \delta(u, v) \in S\}$ where $\delta(u, v) = min\{\pm|u - v| \pmod{n}\}$.
We will denote it by $C(n, S)$.



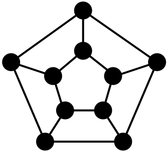

$C(7, \{1, 2\})$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 2 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 4 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 5 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 6 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |

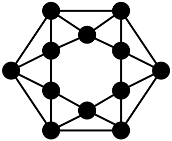
# Some well-known circulant graphs



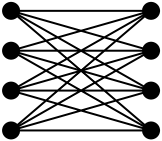
$a)\ S = \{1\}$



$b)\ S = \{2, 5\}$



$c)\ S = \{1, 2\}$

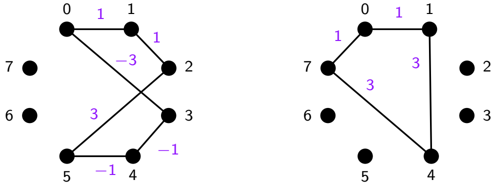


$d)\ S = \{1, 3\}$

Figure: Examples of circulants with their connecting sets

## Definition

For a cycle $C_m = (v_1, \ldots, v_m)$ in a circulant $C(n, \{d_1, \ldots, d_k\})$ we define the *edge-length sequence* as a sequence of integers $(e_1, \ldots, e_m)$ from the set $\{-d_k, \ldots, -d_1, d_1, \ldots, d_k\}$ where $e_i \equiv v_{i+1} - v_i \pmod{n}$ for $i \in \{1, \ldots, m-1\}$ and $e_m \equiv v_1 - v_m \pmod{n}$.




$C_6 = (0, 1, 2, 5, 4, 3)$

$(1, 1, 3, -1, -1, -3)$

$C_4 = (0, 1, 4, 7)$

$(1, 3, 3, 1)$

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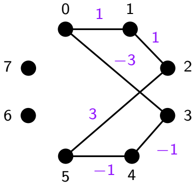


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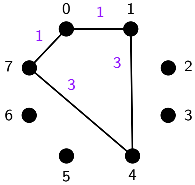


$C_4 = (0, 1, 4, 7)$

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## Definition

We call a cycle $C_m$ a *reversing cycle* if for its corresponding edge-length sequence $(e_1, \ldots, e_m)$ the following holds $e_1 + \ldots + e_m = 0$ in $\mathbb{Z}$. Otherwise, we call $C_m$ a *winding cycle*.

# Alspach's conjecture (1981)

**Conjecture**

Let $n$ be an odd integer, and let $L = (l_1, \ldots, l_t)$ be a list of integers. There exists a decomposition $\{C_{l_1}, \ldots, C_{l_t}\}$ of $K_n$ if and only if

- $3 \leq l_i \leq n$ for each $i \in \{1, \ldots, t\}$ and
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Proven by Bryant, Horsley, and Pettersson in 2014.

They also solved the case where the order of the complete graph is even.

# Bryant and Martin's results

**Theorem (Bryant, Martin) 2009**

Let $n \geq 7$ and let $L = (l_1, \ldots, l_t)$ be a list of integers with $3 \leq l_i \leq 5$ for $i \in \{1, \ldots, t\}$ and $l_1 + \ldots + l_t = 3n$. Then there exists a decomposition $\{C_1, \ldots, C_t\}$ of $C(n, \{1, 2, 3\})$.

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Let $n \geq 5$ and let $l_1, \ldots, l_t$ be a sequence of integers with $3 \leq l_i \leq n$ for $i \in \{1, \ldots, t\}$. There exists a decomposition $\{C_1, \ldots, C_t\}$ of $C(n, \{1, 2\})$ if and only if both of the following conditions hold

1. $l_1 + \ldots + l_t = 2n$ and
2. either
   - $t = 3$ and $\frac{n}{2} \leq l_1, l_2, l_3 \leq n$ or
   - there exists an integer $k \in \{1, \ldots, t\}$ such that $l_k \geq n - t + 1$.

# Cycle decomposition of $C(n, \{1, 3\})$

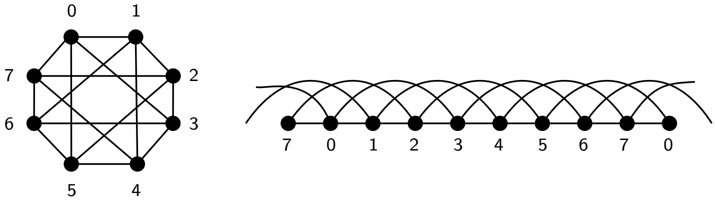


Figure: Equivalent representations of $C(8, \{1, 3\})$

# Cycle decomposition of $C(n, \{1, 3\})$

## Definition

A list $L = (l_1, \dots, l_t)$ of cycle lengths is *admissible* if the following conditions are satisfied:

1. if $l_i$ is even then $4 \leq l_i \leq n$ and otherwise $\lceil \frac{n}{3} \rceil \leq l_i \leq n$
2. $l_1 + \ldots + l_t = 2n$

## Definition

An admissible list $L = (l_1, \dots, l_t)$ is *ordered* when

- if $l_i$ is odd and $l_j$ is even then $i < j$
- if $l_i$ and $l_j$ are of the same parity and $l_i < l_j$ then $i < j$

For $n = 17$ an ordered admissible list could be $L = (7, 9, 4, 6, 8)$.

# Cycle decomposition of $C(n, \{1, 3\})$ - first observations

**Lemma**

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Conclusion: Odd cycles need to be of type winding.

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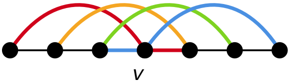
**Lemma**

If there exists a decomposition of $C(n, \{1, 3\})$ into cycles of lengths from the list $L = (l_1, \ldots, l_t)$, then the number of odd lengths in $L$ must be:

- even, since $l_1 + \ldots + l_t = 2n$
- at most 4



# Main result

**Theorem (Meszka, P.)**

Let $n \geq 7$ be an integer and $L = (l_1, \ldots, l_t)$ be an ordered admissible list. Let $t'$ denote the number of odd elements in $L$. There exists a decomposition $\{C_1, \ldots, C_t\}$ of the circulant $C(n, \{1, 3\})$ if and only if one of the following conditions is satisfied:

1. $t' = 0$ and $L \neq (4, \ldots, 4, n)$ when $n \equiv 0 \pmod 8$ and $t = \frac{n}{4} + 1$, or
2. $t' = 2$, $n$ is odd, $t \geq n - \frac{l_2 + 3l_1}{2} + 2$ and moreover $L \neq (\frac{n+4}{3}, n, \frac{2n-4}{3})$ when $n \equiv 5 \pmod 6$ and $t = 3$, or
3. $t' = 4$ and $n$ is odd.

# DECOMPOSITION INTO ODD CYCLES

# The case of 4 odd cycles

Let's assume $n \equiv 3 \pmod 6$ and $L = (l_1, l_2, l_3, l_4)$ - ordered, admissible.

Let $l_i = \frac{n}{3} + m_i$, where $m_i$ is an even integer for $i \in \{1, 2, 3\}$.
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Example:

$$n = 15, \quad L = (5, 7, 9, 9)$$
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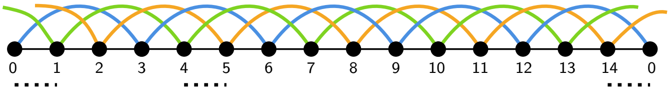
Observation: $m_1 + m_2 + m_3 \leq \frac{n-1}{2}$.

Otherwise, $l_4 \leq \frac{n-1}{2}$ and $l_1 + l_2 + l_3 + l_4 < 2n$.

# From the base case to all possible lengths

Step 1: Constructing the base case, $L_b = (\frac{n}{3}, \frac{n}{3}, \frac{n}{3}, n)$.

Example: $n = 15$ and $L = (5, 7, 9, 9)$



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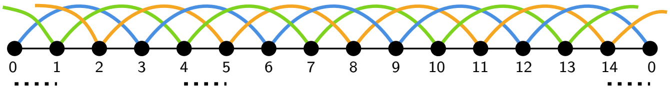
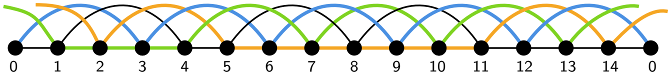



Step 2: Interchanging colors between the Hamiltonian path and shorter paths on proper segments.



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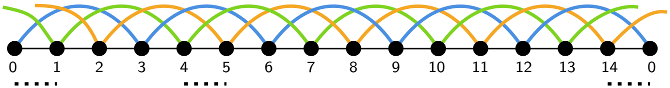


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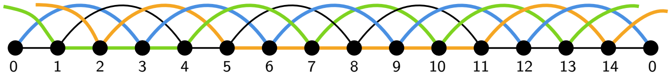


Note: This strategy is possible if $\frac{3}{2}(m_1 + m_2 + m_3) + 3 \leq n$.

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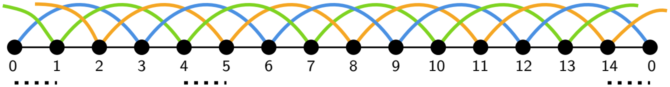


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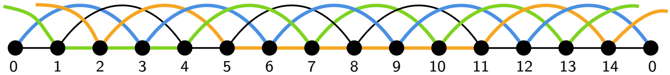
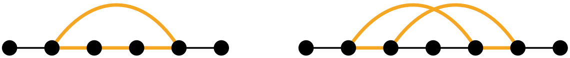



Note: This strategy is possible if $\frac{3}{2}(m_1 + m_2 + m_3) + 3 \leq n$.
And this holds by the previous observation $(m_1 + m_2 + m_3 \leq \frac{n-1}{2})$.

# DECOMPOSITION INTO EVEN CYCLES

# Exception

There is no decomposition when $n \equiv 0 \pmod 8$ and $L = (4, \ldots, 4, n)$.



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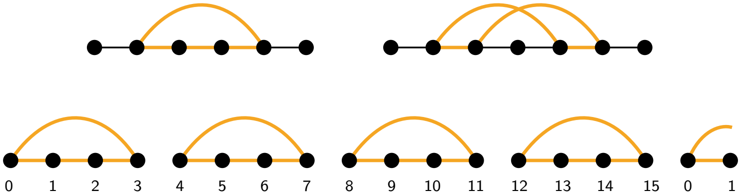

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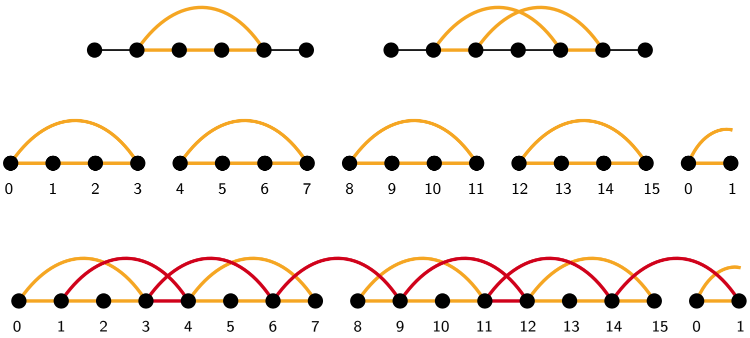
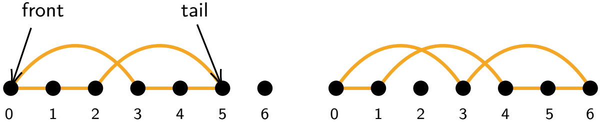


# Assumptions of the algorithm

1. Cycles are of type reversing.
2. The front of each cycle has an even index.

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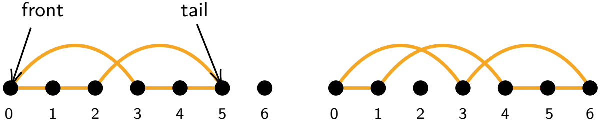


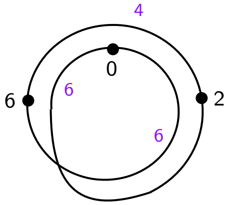

3. Each cycle may omit at most one vertex in between.

**Observation**

A vertex can be a front vertex for only one cycle.

# Finding positions for fronts of cycles - method

Example: $n = 8$ and $L = (4, 6, 6)$



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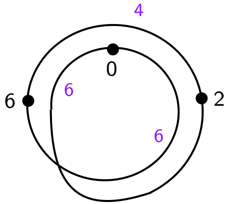



Problem solved for the following base cases:

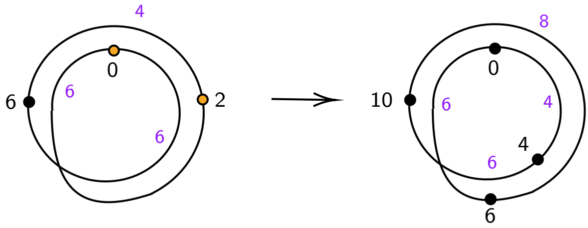
- uniform list $L = (4, 4, 4, 4, 4)$
- two types of lengths differing by 2 $L = (8, 8, 8, 10, 10)$
- three different lengths and $|L| = 3$ $L = (4, 8, 10)$

. . . for any other list we use recursion.

# Finding positions for fronts of cycles - example

$n = 12 \qquad L = (4, 6, 6, 8)$

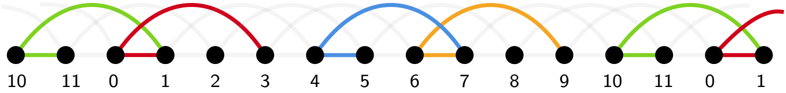
$n' = 8 \qquad L' = (4, 6, 6)$



# Decomposition - example

Decomposition of $C(12, \{1, 3\})$ into $\{C_4, C_6, C_6, C_8\}$.

| front index | 0 | 4 | 10 | 6 |
|---|---|---|---|---|
| cycle length | 4 | 6 | 8 | 6 |



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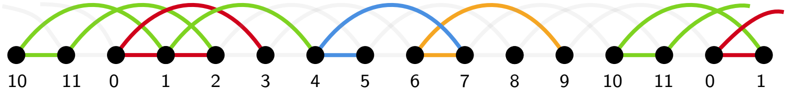

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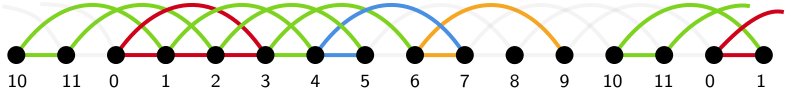

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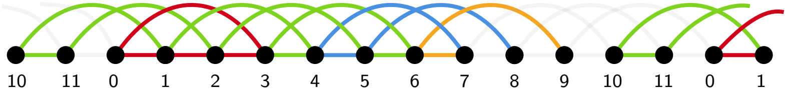

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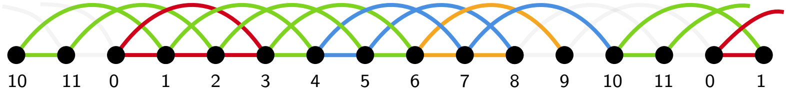

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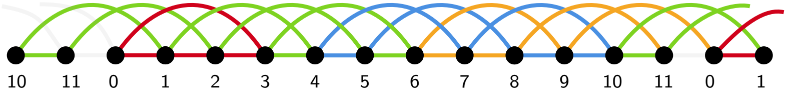
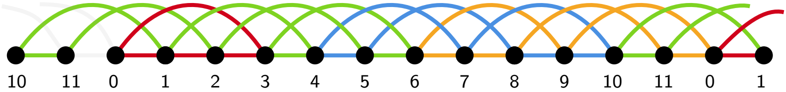


# Decomposition - example

Decomposition of $C(12, \{1, 3\})$ into $\{C_4, C_6, C_6, C_8\}$.

| front index | 0 | 4 | 10 | 6 |
|---|---|---|---|---|
| cycle length | 4 | 6 | 8 | 6 |



MIXED DECOMPOSITION

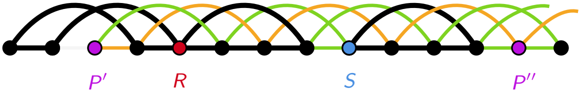
$$t' = 2,\ n \text{ is odd},\ t \geq n - \frac{l_2 + 3l_1}{2} + 2$$

and

$$L \neq (\tfrac{n+4}{3}, n, \tfrac{2n-4}{3}) \text{ when } n \equiv 5 \pmod 6 \text{ and } t = 3$$

## Observation

A pair of even cycles may have at most one vertex in common.

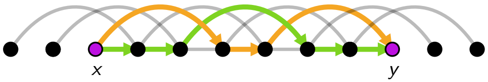


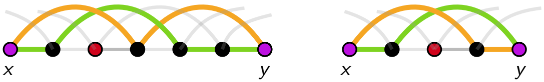

We distinguish the following subsets of the set of vertices:

- $P = P' \cup P''$ - vertices that appear in both odd cycles, $v \in P'$ iif $(v-1)$ and $(v+1)$ are in the same even cycle
- $R$ - vertices that are both a front and a tail of an even cycle
- S - verices that are only a front of an even cycle

We denote the sizes of these sets by lowercase letters, f.ex. $|P'| = p'$.

# A note on length measurement




| Type of segment | $P' - P'$ | $P' - S$ | $S - P'$ | $S - S$ |
|---|---|---|---|---|
| Length differences | 0 / 2 | 3 | 1 | 0 / 2 |

# Necessary condition for $L = (l_1, l_2, \ldots, l_t)$

1. $p' + s \geq \frac{l_2 - l_1}{2}$
2. By the handshaking lemma:

$$2(l_1 + l_2) = 2(n - r - p) + 4p \implies r = n + p - (l_1 + l_2)$$

$$\begin{aligned} t - 2 = r + s &= n + p - (l_1 + l_2) + s \\ &\geq n - (l_1 + l_2) + p' + s \\ &\geq n - (l_1 + l_2) + \frac{l_2 - l_1}{2} \\ &= n - \frac{n_2 + 3n_1}{2} \end{aligned}$$

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$$\implies t \geq n - \frac{n_2 + 3n_1}{2} + 2$$

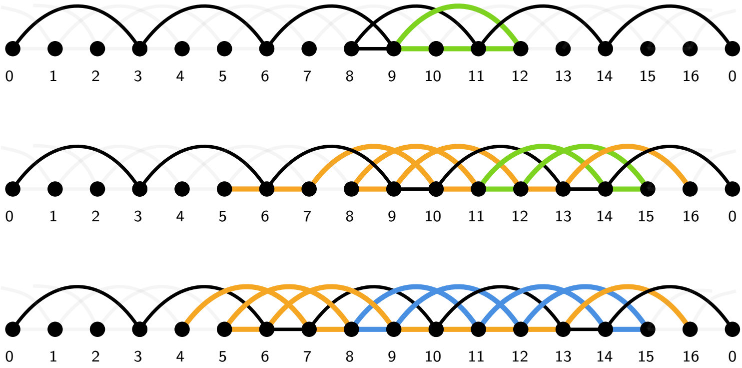
**Remark**

This necessary condition is also sufficient when $l_2 < n$.

# Hamiltonian odd cycle - exception

Exception: $n \equiv 5 \pmod 6$, $L = (\frac{n+4}{3}, n, \frac{2n-4}{3})$

Example: $n = 17$, $L = (7, 17, 10)$



# Further research

1. Cycle decomposition of $C(n, \{1, k\})$, where $k \notin \{2, 3\}$
2. 2-factorization of circulant graphs

Thank you for your attention!