# Hamilton compression

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## 12th PhD Summer School in Discrete Mathematics
## September 7 - 13, 2025
## UP FAMNIT, Koper, Slovenia

- Minicourse 1: A short course in spectral graph theory and distance-regular graphs

  Lecturer: Sebastian Cioaba, University of Delaware, USA.
- Minicourse 2: Container method in combinatorics

  Lecturer: Rajko Nenadov, University of Auckland, New Zealand

Confirmed Invited Speakers

- Aida Abiad, Eindhoven University of Technology, Netherlands
- Bence Csajbók, ELTE Eötvös Loránd University, Hungary
- Miguel Angel Fiol, Polytechnic University of Catalunya, Spain
- Daniel Král', Leipzig University, Germany
- Luke Morgan, The University of Western Australia, Australia

11th Slovenian Conference on Graph Theory
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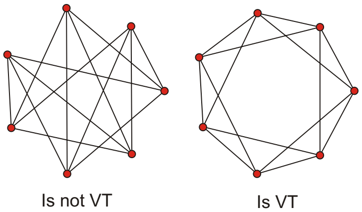
## Lovász's problem

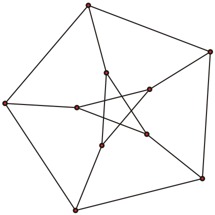
Does every connected vertex-transitive graph have a Hamilton path?

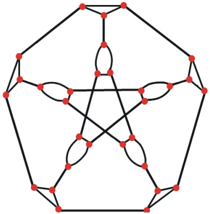
- A Hamilton path is a spanning path in a graph.
- A Hamilton cycle is a simple cycle containing all vertices of the graph.
- A graph is vertex-transitive if its automorphism group acts transitively on vertices.



Only four connected vertex-transitive graphs (having at least three vertices) not having a Hamilton cycle are known to exist:

- the Petersen graph,
- the Coxeter graph,
- and the two graphs obtained from them by replacing each vertex with a triangle.





None of these four graphs is a Cayley graph. This has led to a folklore conjecture that every connected Cayley graph has a Hamilton cycle.

**Conjecture (Thomassen, '91)**

There exist only finitely many connected vertex-transitive graphs without a Hamilton cycle.

**Conjecture (Babai, '79)**

There exist infinitely many such graphs.

Hamilton cycles (paths) are known to exist in these cases (where $p$ and $q$ are primes):

- VTG of order $p$, $2p$, $3p$, $4p$, $5p$, $6p$, $10p$, $2p^2$, $p^k$ (for $k \leq 4$), $pq$ (Alspach, Chen, Du, Marušič, Parsons, Šparl, Zhang, KK, Yu, Luo, etc.);
- VTG of order $2pq$ with primitive aut. group (Du, Tian, Yu, 2022);
- VTG having groups with a cyclic commutator subgroup of order $p^k$ (Durenberger, Gavlas, Keating, Marušič, Morris, Morris-Witte, etc.).
- Cayley graphs of order $p^k$ (Morris-Witte).
- Cayley graphs of order less than 144 (Ghaderpour, KK, Marušič, Morris, Morris, Šparl, Wilk)
- Cayley graphs of odd order groups with a cyclic commutator subgroup of order $p^k q^l$ (Morris-Witte);
- Cubic Cayley graphs $Cay(G,S)$, where $S = \{x, y\}$ and $x^n = 1$, $y^2 = 1$ and $(xy)^3 = 1$;
- Cubic Cayley graphs $Cay(G,S)$, where $S = \{a, b, c\}$ and $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, (ac)^s = 1, (bc)^t = 1, \ldots \rangle$;
- and in some other cases.

In short: the problem is still open.

- N. Draganić, R. Montgomery, D. M. Correia, A. Pokrovskiy, B. Sudakov, Hamiltonicity of expanders: optimal bounds and applications, arXiv:2402.06603v2, April 2024:

**arXiv:2402.06603v2, Theorem 1.8**

Let $C$ be a sufficiently large constant. Let $G$ be a group of order $n$ and $d \geq C \log n$. If $S \subseteq G$ is a set of size $d$ chosen uniformly at random, then, with high probability, $Cay(G, S)$ is hamiltonian.

- S. Bonvicini, T. Pisanski and A. Žitnik, All rose window graphs are hamiltonian, arXiv:2504.16205, April 2025:

**arXiv:2504.16205, Theorem 1**

Every connected generalized rose window graph is hamiltonian.

# Rose Window Graphs

Introduced by Steve Wilson.

**Definition**

Given natural numbers $n \geq 3$ and $1 \leq a, r \leq n-1$, the rose window graph $R_n(a, r)$ is a quartic graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{y_i y_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_i \mid i \in \mathbb{Z}_n\} \cup \{x_{i+a}, y_i \mid i \in \mathbb{Z}_n\}$.

The Rose Window graph $R_n(a, r)$ contains, as a spanning subgraph, the generalized Petersen graph $GP(n, r)$.

**Definition**

Given natural numbers $n \geq 3$ and $1 \leq a, r \leq n-1$, the generalized rose window graph $R_n(a, r, s)$ is a quartic graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+s} \mid i \in \mathbb{Z}_n\} \cup \{y_i y_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_i \mid i \in \mathbb{Z}_n\} \cup \{x_{i+a}, y_i \mid i \in \mathbb{Z}_n\}$.

Introduced by Gregor, Merino in Mütze in 2022 (published in Annals of Combinatorics in 2023).

### $k$-symmetric Hamilton cycle

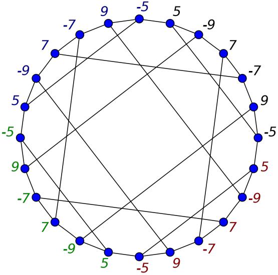
A Hamilton cycle $C = v_0, v_1, \ldots, v_{n-1}, v_0$ in a graph $X$ is $k$-symmetric (it admits a rotational symmetry of order $k$) if there exists an automorphism $\alpha$ of $X$ such that $\alpha(v_i) = v_{i+n/k}$ for each $i \in \mathbb{Z}_n$.

This means that $Aut(C) \cap Aut(X)$ contains a cyclic group of order $k$.

### Compression factor

The maximum $k$ for which the Hamilton cycle $C$ of $X$ is $k$-symmetric is the compression factor of $C$, denoted by $\kappa(X, C)$.

The generalized Petersen graph $\mathrm{GP}(12, 5)$.




$[5, -9, 7, -7, 9, -5]^4$

There is a natural connection of the compression factor to the so-called LCF notation for cubic hamiltonian graphs, which describes a cubic hamiltonian graph $X$ via one of its Hamilton cycles $C = v_0, v_1, \ldots, v_{n-1}$ through a sequence $[d_0, d_1, \ldots, d_{n-1}]$, where $d_i := j - i \pmod{n}$ with $v_j$ being the third neighbour of $v_i$ (different from $v_{i\pm1}$). Also, $-n/2 < d_i \leq n/2$ and $d_i \notin \{0, \pm 1\}$.

## Hamilton compression

The Hamilton compression $\kappa(X)$ of $X$ is the maximum of $\kappa(X, C)$ where $C$ runs over all Hamilton cycles in $X$.

$$\kappa(X) := \max\{\kappa(X, C) \mid C \text{ is a Hamilton cycle in } X\}.$$

If $X$ has no Hamilton cycle, we let $\kappa(X) := 0$.
When $\kappa(X) = 1$ we say that $X$ has a trivial Hamilton compression.

While for the Petersen graph $\mathrm{GP}(5,2)$ we have $\kappa(\mathrm{GP}(5,2)) = 0$, the complement of $\mathrm{GP}(5,2)$ has 1-symmetric and 5-symmetric Hamilton cycles, and thus

$$\kappa(\overline{\mathrm{GP}(5,2)}) = 5$$

**Gregor, Merino and Mütze, Ann. Comb., 2023**

They investigate the Hamilton compression of four different families of vertex-transitive graphs (hypercubes, Johnson graphs, permutahedra and Cayley graphs of abelian groups). In several cases they determine their Hamilton compression exactly, and in other cases they provide close lower and upper bounds.

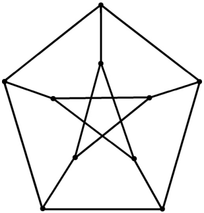
The concept of rotational symmetry of Hamilton cycles is directly linked to a widely used method for constructing Hamilton cycles in connected vertex-transitive graphs, the so-called Lifting Cycle Technique.

The concept of rotational symmetry of Hamilton cycles is directly linked to a widely used method for constructing Hamilton cycles in connected vertex-transitive graphs, the so-called Lifting Cycle Technique.

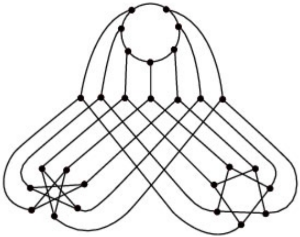
This approach is based on quotienting the graph with respect to the set of orbits of a suitable semiregular automorphism. Provided the quotient graph contains a Hamilton cycle it is sometimes possible to lift this cycle to construct a Hamilton cycle in the original graph. This construction is made easier when the semiregular automorphism is of prime order.

An automorphism is $(m, n)$-semiregular if it has $m$ cycles of lenght $n$ in its cyclic decomposition.

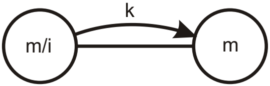
The Petersen and the Coxeter graph ($G = PSL(2, 7)$ and $H = S_3$). Both are non-hamiltonian.



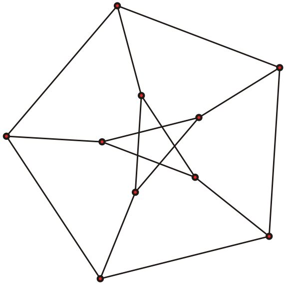
Has a (2,5)-semiregular automorphism



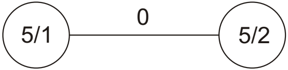
Has a (4,7)-semiregular automorphism



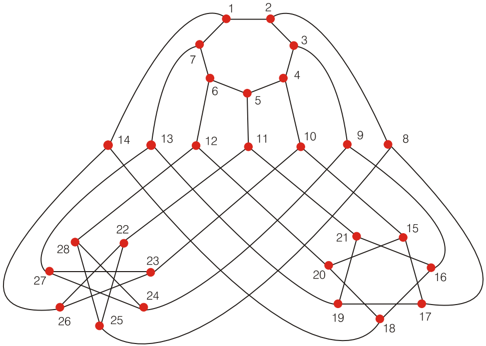

- Each circle denotes $m$ vertices: $v_0, \ldots, v_m$ and $u_0, \ldots, u_m$.
- $m/i$ denotes $v_j \sim v_{j+i}$ with addition modulo $m$.
- $m$ denotes $\overline{K_m}$.
- $m/i, l, \ldots$
- The undirected edge denotes $v_j \sim u_j$ for each $j$.
- The directed edge with label $k$ denotes $v_j \sim u_{j+k}$ .

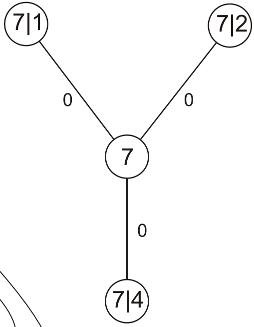


a $(2,5)$-semiregular automorphism $\rho = (1\,2\,3\,4\,5)(1'\,2'\,3'\,4'\,5')$

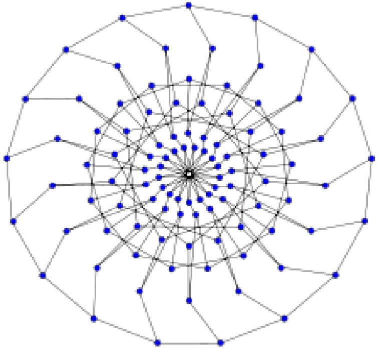


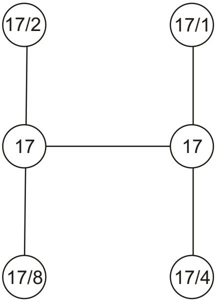
# Example - the Coxeter graph



# Example - the Biggs-Smith graph





**Polycirculant conjecture (Marušič, '81; for 2-closed groups, Klin, '96)**

Does every vertex-transitive (di)graph have a semiregular automorphism?

The problem has been only partially solved. Many authors: Cameron, Dobson, Giudici, Hujdurović, Jones, Kantor, Klin, Kovács, KK, Li, Nowitz, Malnič, Scapellato, Spiga, Šparl, Verret, Xu, ...

The 2-closure $G^{(2)}$ of a group $G$ is the group of all permutations of $\Omega$ which fix setwise each orbit of $G$ on $\Omega \times \Omega$.

A permutation group $G$ is 2-closed if $G = G^{(2)}$.

- All transitive permutation groups of degree $p^k$ or $mp$, for some prime $p$ and $m < p$, have SE of order $p$ (Marušič, '81).
- All cubic VTG have SA (Marušič, Scapellato, '93).

  There exists one of order $> 2$ (Cameron-Sheehan-Spiga, '08).

  There exists a function $f(n)$ such that $f(n) \to \infty$ as $n \to \infty$ such that a cubic VTG has a semiregular subgroup of order $> f(n)$ (Li, '08).
- All VTD of order $2p^2$ have SA of order $p$ (Marušič, Scapellato, '93).

**Primitive, Quasiprimitive and Biquasiprimitive groups**

Let $G \leq Sym(\Omega)$ be transitive. Then

- $G$ is primitive if there are no nontrivial partitions of preserved by $G$.
- $G$ is quasiprimitive if all nontrivial normal subgroup of $G$ are transitive.
- $G$ is biquasiprimitive if $G$ is not quasiprimitive and all nontrivial normal subgroups have at most two orbits.

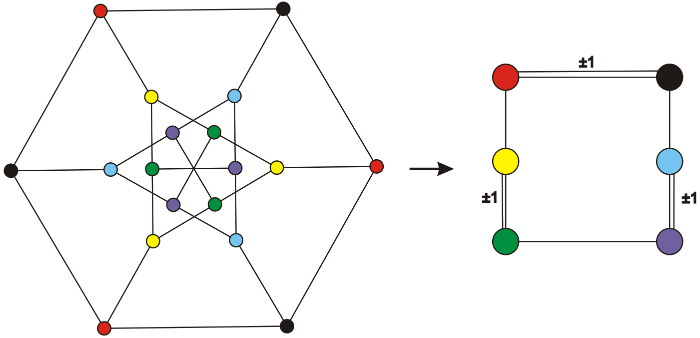
- All vertex-primitive graphs have SA (Giudici, '03).
- All vertex-quasiprimitive graphs have SA (Giudici, '03).
- All vertex-transitive bipartite graphs where only system of imprimitivity is the bipartition, have SA (Giudici, Xu, '07).

- Every 2-arc-transitive graph has SA (Xu, '07).
- Every ATG of prime valency has SA (Xu, '07).
- All quartic VTG have SA (Dobson, Malnič, Marušič, Nowitz, '07).
- All VTG of valency $p+1$ admitting a transitive $\{2,p\}$-group for $p$ odd have SA (Dobson, Malnič, Marušič, Nowitz, '07).
- There are no elusive 2-closed groups of square-free degree (Dobson, Malnič, Marušič, Nowitz, '07).
- All ATG with valency $pq$, $p$, $q$ primes, such that Aut(X) has a nonabelian minimal normal subgroup $N$ with at least 3 vertex orbits, have SA (Xu, '08).
- Every VTG, edge-primitive graph has SA (Giudici, Li, '09).
- All distance-transitive graphs have SA (KK, Šparl, '10).

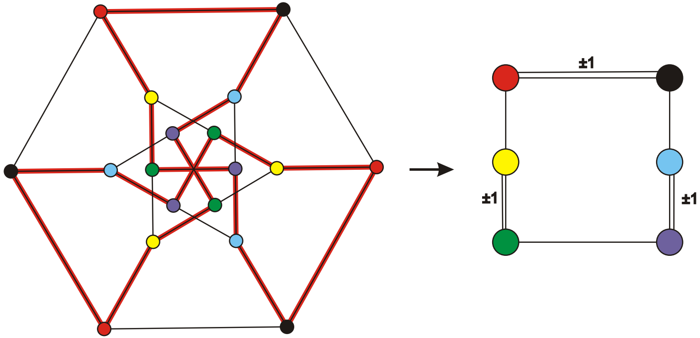
Why is it useful to know whether a vertex-transitive graph admits a semiregular automorphism?

It allows a quotienting with respect to semiregular automorphisms, and when the latter is wisely chosen, properties of the original graph can be analyzed via its quotient.

The Pappus graph



# The Pappus graph




$$\kappa(X) = 3$$

### Lifting Hamilton cycle approach

- Let $X$ be a VTG.
- Find a semiregular automorphism $\rho$ (of prime order).
- Take the quotient graph $X_\rho$ with respect to orbits of $\rho$.
- Find a Hamilton cycle in $X_\rho$ (with non-zero voltage).
- Lift this cycle to a Hamilton cycle in $X$.

**Theorem (Du, KK, Marušič, '21)**

With the exception of the Petersen graph, a connected vertex-transitive graph of order $pq$, where $p$ and $q$ are primes, contains a Hamilton cycle.


- S. F. Du, K. Kutnar and D. Marušič, Resolving the hamiltonian problem for vertex-transitive graphs of order a product of two primes, *Combinatorica* **41** (2021), 507–543.

**Theorem (Scapellato, Marušič, 1994)**

A connected vertex-transitive graph of order $pq$, where $p$ and $q$ are odd primes and $p > q$, must be one of the following:

(i) a metacirculant (Subclass 1),

(ii) a Fermat graph (Subclass 2),

(iii) a generalized orbital graph associated with one of the groups consisting of 'genuinely' primitive groups of degree $pq$ (Subclass 3).

- K. Kutnar, D. Marušič and A. S. Razafimahatratra, Hamiltonicity of certain vertex-transitive graphs revisited, *Discrete Math.* **348** (2025), 114350.
- K. Kutnar, D. Marušič and A. S. Razafimahatratra, Infinite families of vertex-transitive graphs with prescribed Hamilton compression, *Ann. Combin.* **28** (2024), 1243-1255.

**Theorem (KK, Marušič, Razafimahatratra, 2024)**

Let $p > q$ be primes and let $X$ be a $(q,p)$-metacirculant which is not a circulant. Then the Hamilton compression of $X$ is as follows:

(i) $\kappa(X) = 0$ if $X$ is isomorphic to the Petersen graph;
(ii) $\kappa(X) = 1$ if $X$ is a non-Cayley graph and $\tilde{X}$ is a disconnected graph;
(iii) $\kappa(X) = q$ if $X$ is a Cayley graph and $\tilde{X}$ is a disconnected graph;
(iv) $\kappa(X) = p$ if $\tilde{X}$ is a connected graph.

Given an $(m,n)$-metacirculant $X$ with an $(m,n)$-semiregular automorphism $\rho$ we let $\tilde{X}(\rho)$ denote the subgraph obtained from $X$ by removing all the edges joining two vertices from the same orbit of $\rho$.

**Theorem (KK, Marušič, Razafimahatratra, 2025)**

Let $X$ be a vertex-primitive graph of order $pq$, where $p > q$ are distinct primes. Then $\kappa(X) \in \{p, q\}$.

**Theorem (KK, Marušič, Razafimahatratra, 2025)**

Let $p = 1 + 2^{2^e}$, $e > 2$, be a prime and let $q = 3$. Then for the Fermat graph $X$ of order $3p$ we have $\kappa(X) = 1 + 2^{2^e}$.

**P. Gregor, A. Merino and T. Mütze, The Hamilton compression of highly symmetric graphs, Ann. Comb., 2023**

Are there infinitely many vertex-transitive graphs $X$ with $\kappa(X) = k$, for each fixed integer $k$?

**Problem (small amendment)**

Given a positive integer $k$, are there infinitely many vertex-transitive non-Cayley graphs $X$ with $\kappa(X) = k$, and similarly, are there infinitely many Cayley graphs $X$ with $\kappa(X) = k$?

- An infinite family of Cayley graphs with Hamilton compression equal to 1 was given in the paper of Gregor, Merino and Mütze.
- In the case of Cayley graphs the question is completely resolved in

> K. Kutnar, D. Marušič and A. S. Razafimahatratra, Infinite families of vertex-transitive graphs with prescribed Hamilton compression, *Ann. Combin.* **28** (2024), 1243-1255.

with a construction of Cayley graphs of semidirect products $\mathbb{Z}_p \rtimes \mathbb{Z}_k$ where $p$ is a prime and $k \geq 2$ a divisor of $p-1$ with Hamilton compression equal to $k$.

**Dirichlet prime number theorem (1937)**

If $a$ and $b$ are relatively prime positive integers, then there are infinitely many primes of the form $a + jb$ with $j \in \mathbb{Z}$.

Letting $a = 1$ and $b = k$ there exist infinitely many primes $p$ such that $k$ divides $p - 1$.

In

> K. Kutnar, D. Marušič and A. S. Razafimahatratra, Infinite families of vertex-transitive graphs with prescribed Hamilton compression, *Ann. Combin.* **28** (2024), 1243-1255.

infinite families of non-Cayley vertex-transitive graphs with Hamilton compression equal to 1 are also given.

Thank you!