Geometry of binary simplex codes and symmetric block designs

Krzysztof Petelczyc

(Joint work with Mark Pankov and Mariusz Żynel)

University of Białystok, Poland

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Geometry with projective lines over $\mathbb{F}_2$

Consider the vector space $\mathbb{F}_2^n$. Set $[n] = \{1, \ldots, n\}$.

- the points of $\mathrm{PG}(n-1, 2) \equiv$ the non-empty subsets of $[n]$,

$$\text{e.g. } (1,1,1,1,0,0,0) \equiv \{1,2,3,4\} \text{ for } n = 7$$

- the third point on the line containing X and Y is $X \triangle Y$,

$$\text{e.g. } (1,1,1,1,0,0,0) + (1,1,0,0,0,1,1) = (0,0,1,1,0,1,1)$$
$$\{1,2,3,4\} \triangle \{1,2,6,7\} = \{3,4,6,7\}$$

- the set of all t-element subsets of $[n]$ contains lines of $\mathrm{PG}(n-1,2)$ $\iff$ $t = 2m$ and $n \geq 3m$

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M. Pankov, K.P., M. Żynel, *Point-line geometries related to binary equidistant codes*, J. Comb. Theory Ser. A, **210** (2025), 1-30.

For $n \geq 3m$ we introduce the point-line geometry:

- points are $2m$-element subsets of the set $[n]$,
- lines are the lines of $\mathrm{PG}(n-1,2)$ formed by these subsets,
- two distinct points X and Y are collinear iff $|X \cap Y| = m$.

Denote this geometry by $\mathcal{P}_m([n])$ or simply by $\mathcal{P}_m(n)$.

Maximal singular subspaces of $\mathcal{P}_m(n)$ correspond to maximal equidistant binary linear codes of length n and Hamming weight $2m$.

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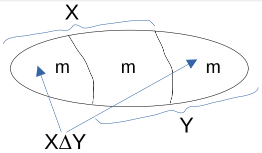

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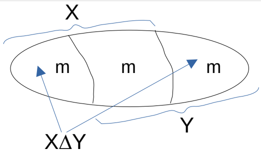

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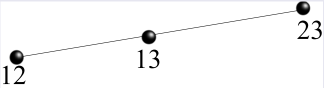
Geometry $\mathcal{P}_m(n)$ – examples

$n = 3,\ m = 1$

A line of size 3



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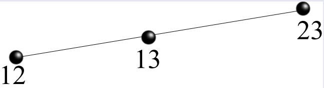

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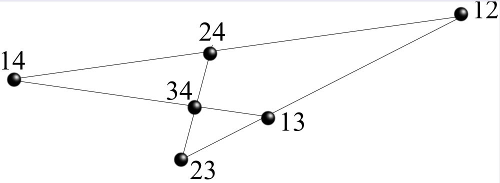

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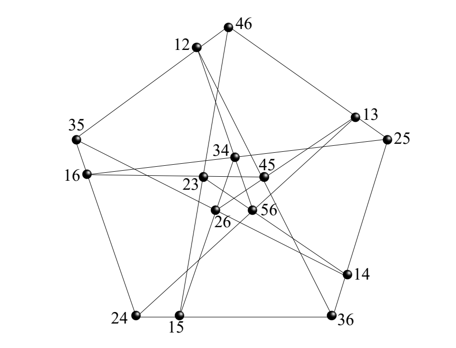
$n = 6,\ m = 2$

The Cremona-Richmond configuration $= \mathrm{GQ}(2,2)$



$n = 2^k - 1,\ m = 2^{k-2}$

Every maximal singular subspace of $\mathcal{P}_m(n)$ is a projective space $\mathrm{PG}(k-1, 2)$ corresponding to a binary simplex code of dimension k.

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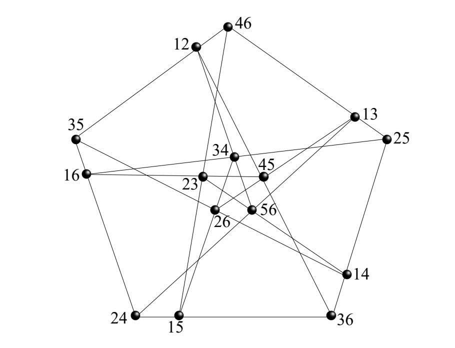

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Every maximal singular subspace of $\mathcal{P}_m(n)$ is a projective space $\mathrm{PG}(k-1,2)$ corresponding to a binary simplex code of dimension k.

The collinearity graph of $\mathcal{P}_m(n)$

- The collinearity graph of $\mathcal{P}_m(n)$ is a simple graph whose vetrices are the points of $\mathcal{P}_m(n)$ and two vertices are adjacent only if they determine a line.
- A subgraph of the collinearity graph is called a clique if it is a complete graph itself.
- A clique which is not contained in any other clique is called a maximal clique.

If $n \geq 3m+1$ and $m \neq 2$, then the collinearity graph of $\mathcal{P}_m(n)$ contains maximal cliques different from maximal singular subspaces.

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Maximal cliques vs symmetric block designs

From now on, for a certain integer $k \geq 2$, we assume that

$$m = 2^{k-2} \quad \text{and} \quad n = 4m - 1 = 2^k - 1.$$

Then every maximal singular subspace of $\mathcal{P}_m(n)$

- corresponds to a binary simplex code of dimension k,
- is a maximal clique of the collinearity graph of $\mathcal{P}_m(n)$,
- is isomorphic to $\mathrm{PG}(k-1, 2)$ and contains $2^k - 1 = n$ elements.

If $\mathcal{C}$ is a maximal clique of $\mathcal{P}_m(n)$ and $|\mathcal{C}| = n$ then $\mathcal{C}$ determines a symmetric $(n, 2m, m)$-design whose

- points are elements of $[n]$,
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Hyperplane complements in $\mathrm{PG}(k-1,2)$

- A hyperplane H of $\mathrm{PG}(k-1,2)$ contains $2^{k-1}-1$ points.
- Its complement H^c contains $2^k-1-(2^{k-1}-1)=2^{k-1}=2m$ points.
- If $H_1 \neq H_2$ the intersection $H_1^c \cap H_2^c$ consists of $2^{k-2}=m$ points.

The design of points and hyperplane complements of $\mathrm{PG}(k-1,2)$ is a symmetric $(n, 2m, m)$-design.

There is the unique hyperplane $H_3 \neq H_1, H_2$ such that $H_1 \cap H_2 \subset H_3$, so

- $H_1 \cup H_2 \cup H_3$ contains all points of $\mathrm{PG}(k-1,2)$,
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Maximal centered cliques

We say that a maximal clique $\mathcal{C}$ is centered if there is $O \in \mathcal{C}$ such that for every $C \in \mathcal{C} \setminus \{O\}$ the line joining C and O is contained in $\mathcal{C}$.
The point O is said to be a center point of $\mathcal{C}$.

- take $O \subset [n]$, $|O| = 2m$, and $[n] \setminus O = O^c$, $|O^c| = 2m - 1$
- consider the geometry $\mathcal{P}_{\frac{m}{2}}(O^c)$
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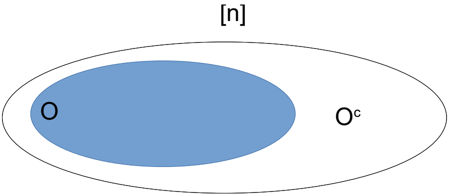

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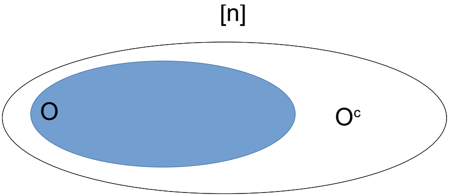

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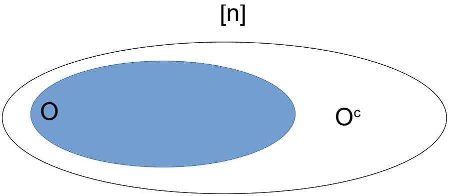

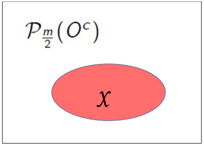

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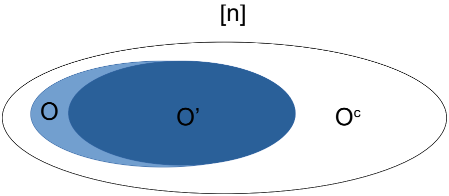
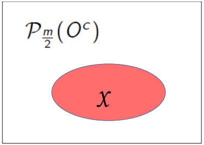
- take $O' \subset O$, $|O'| = 2m - 1$ and consider the geometry $\mathcal{P}_{\frac{m}{2}}(O')$





Maximal centered cliques

We say that a maximal clique $\mathcal{C}$ is centered if there is $O \in \mathcal{C}$ such that for every $C \in \mathcal{C} \setminus \{O\}$ the line joining C and O is contained in $\mathcal{C}$.
The point O is said to be a center point of $\mathcal{C}$.

- take $O \subset [n]$, $|O| = 2m$, and $[n] \setminus O = O^c$, $|O^c| = 2m - 1$
- consider the geometry $\mathcal{P}_{\frac{m}{2}}(O^c)$
- take any maximal clique $\mathcal{X}$ of $\mathcal{P}_{\frac{m}{2}}(O^c)$ such that $|\mathcal{X}| = 2m - 1$
- take $O' \subset O$, $|O'| = 2m - 1$ and consider the geometry $\mathcal{P}_{\frac{m}{2}}(O')$
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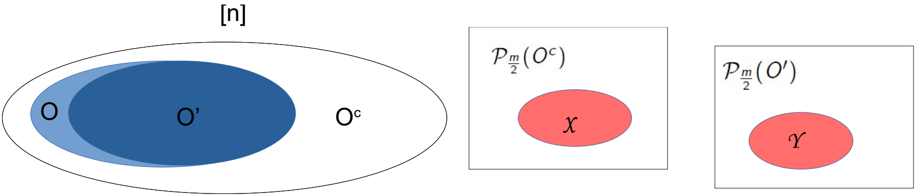

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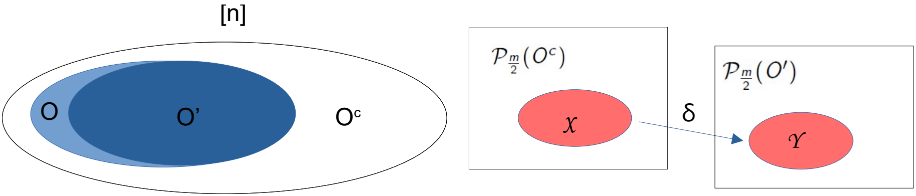

Maximal centered cliques

The result of the construction

The set

$$\{O\} \cup \{X \cup \delta(X) \colon X \in \mathcal{X}\} \cup \{X \cup (O \setminus \delta(X)) \colon X \in \mathcal{X}\}$$

is a maximal centered clique of the collinearity graph of $\mathcal{P}_m(n)$ consisting of $n = 4m - 1$ elements. We denote this clique by $\mathcal{X} \#_\delta \mathcal{Y}$.

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Let $\mathcal{C}$ be a centered maximal n-element clique of the collinearity graph of $\mathcal{P}_m(n)$ and let O be a center point of $\mathcal{C}$.

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Maximal centered cliques of $\mathcal{P}_4(15)$

Let $k = 4$, so $m = 2^{4-2} = 4$, $n = 2^4 - 1 = 15$.

- $\mathcal{P}_{\frac{m}{2}}(2m-1) = \mathcal{P}_2(7)$ is a rank 3 polar space
- every maximal clique of $\mathcal{P}_2(7)$ is a Fano plane
- if $\mathcal{X} \#_\delta \mathcal{Y}$ is a centered maximal clique of $\mathcal{P}_4(15)$, then $\mathcal{X}, \mathcal{Y}$ are Fano planes

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be Fano planes.

- Bijections $\delta, \delta' \colon \mathcal{F}_1 \to \mathcal{F}_2$ are said to be equivalent if there are automorphisms $g_i \colon \mathcal{F}_i \to \mathcal{F}_i$, $i = 1, 2$, such that

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- The index $\mathrm{ind}(\delta)$ is the number of lines which go to lines under δ.

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Bijections between Fano planes – the classification

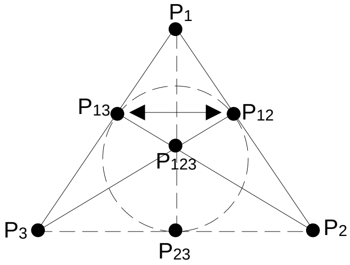

Figure: The case when δ is of index 3

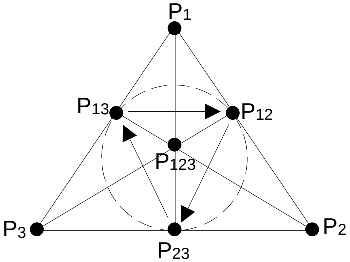

Figure: The case when δ is of index 1

Proposition

Two bijections between $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if and only if they are of the same index. There are precisely four classes of equivalence and the corresponding values of the index are $0, 1, 3, 7$.

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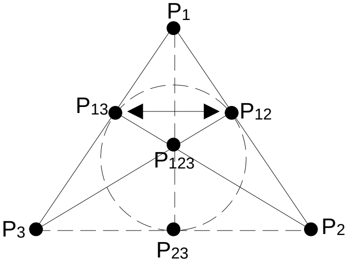

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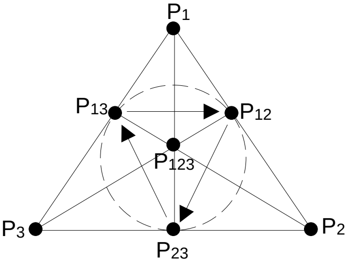

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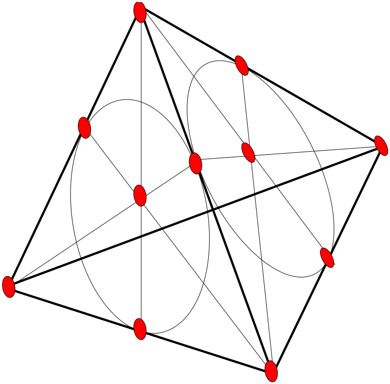
Centered maximal cliques of $\mathcal{P}_4(15)$ – the classification

Theorem

If $\mathcal{C}$ is a centered maximal 15-element clique of the collinearity graph of $\mathcal{P}_4(15)$, then one of the following possibilities is realized:

(C1) *$\mathcal{C}$ is a solid and every point of $\mathcal{C}$ is a center point,*



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If $\mathcal{C}$ is a centered maximal 15-element clique of the collinearity graph of $\mathcal{P}_4(15)$, then one of the following possibilities is realized:

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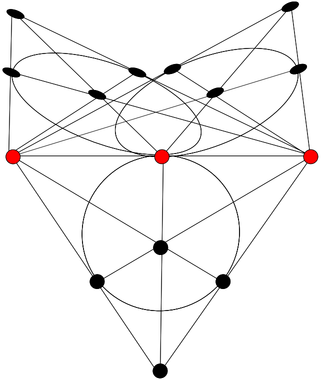

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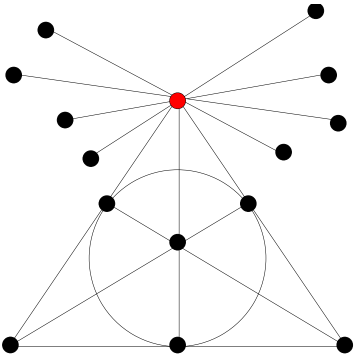

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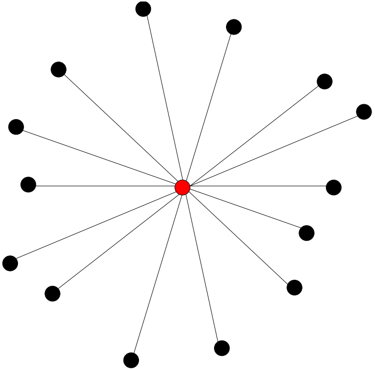

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M. Pankov, K.P., M. Żynel, *Symmetric* $(15, 8, 4)$*-designs in terms of the geometry of binary simplex codes of dimension 4*, Des. Codes Cryptogr. (2025), DOI: 10.1007/s10623-025-01570-7.

Symmetric $(15, 8, 4)$-designs

There are five non-isomorphic symmetric $(15, 8, 4)$-designs.

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There is a non-centered maximal 15-element clique in the collinearity graph of $\mathcal{P}_4(15)$ which is the union of a maximal singular subspace of $\mathcal{P}_4(15)$ with a plane deleted and a plane disjoint with this subspace.

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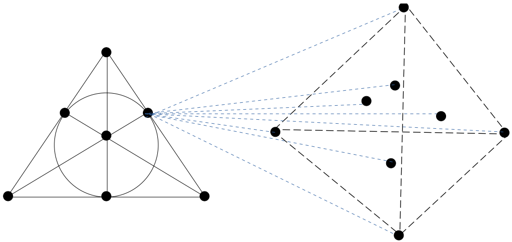

The automorphism group action

- There is no automorphism of the non-centered maximal clique sending points of $\mathcal{S} \setminus \mathcal{F}'$ to points of the plane $\mathcal{F}$.
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Lemma

A symmetric $(15, 8, 4)$-design corresponding to a clique of type $(\mathrm{C1})$ is the unique symmetric $(15, 8, 4)$-design admitting automorphisms acting transitively on the set of blocks.

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That's all Folks!