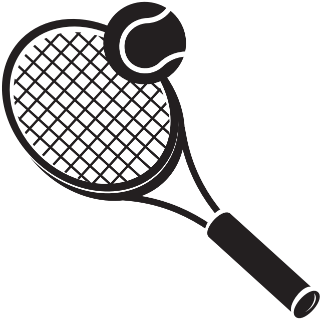
Designs of Perfect Matchings

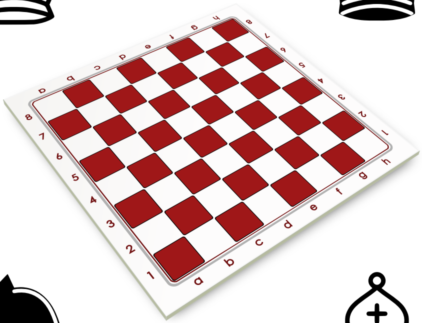
Lukas Klawuhn

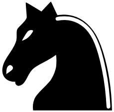
Paderborn University

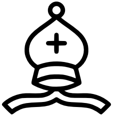
05 June 2025

Games!









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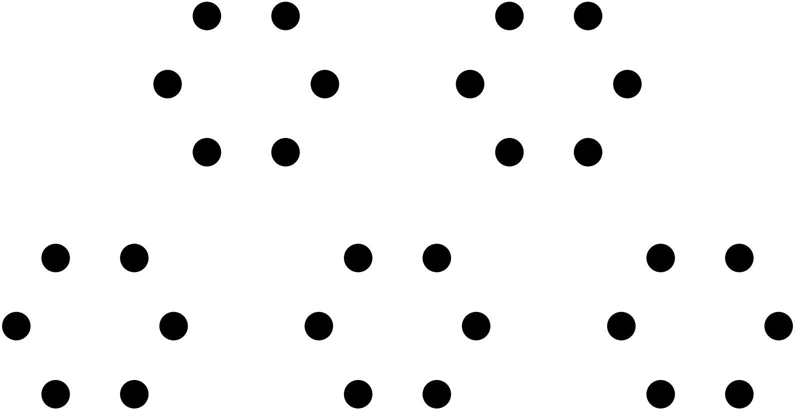

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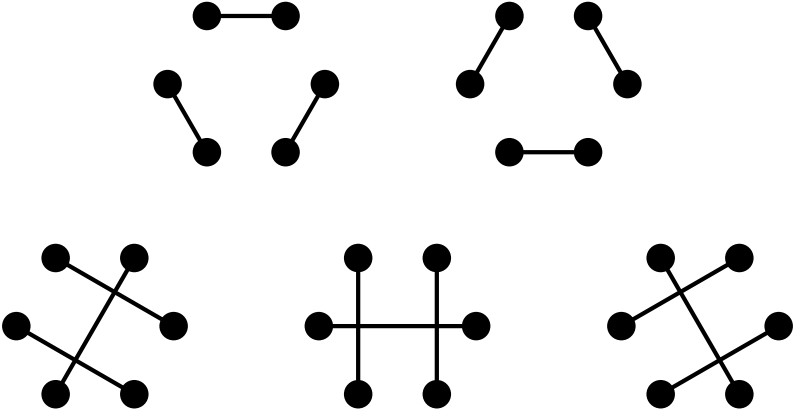

Hyperfactorisations

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Generalisation:

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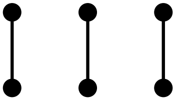
- pair of disjoint subsets $\longrightarrow$ t disjoint subsets

λ-factorisations

t disjoint edges



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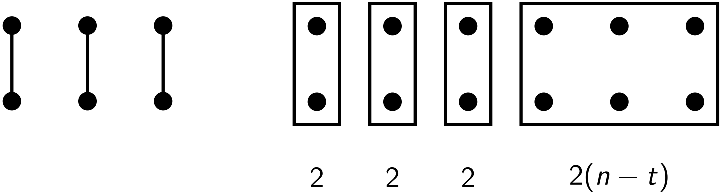
t disjoint edges $\longrightarrow$ set partition of shape $(2(n-t), 2, 2, \ldots, 2)$



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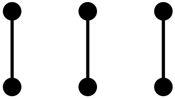

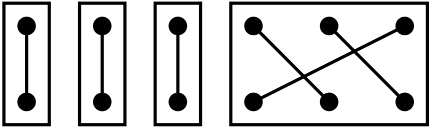

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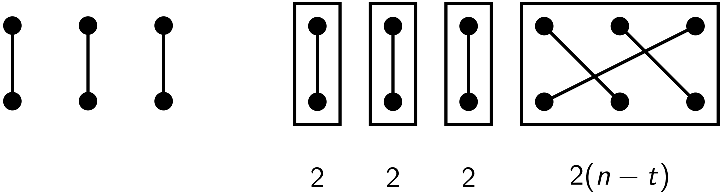

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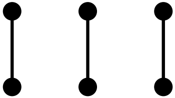

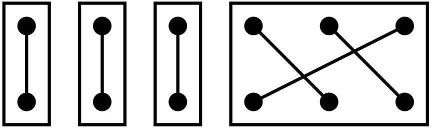

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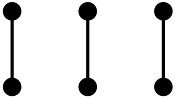

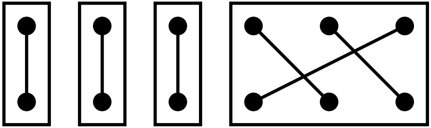

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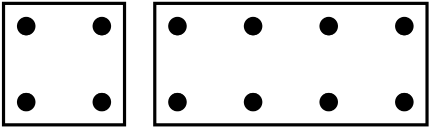

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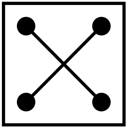

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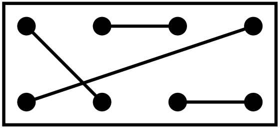

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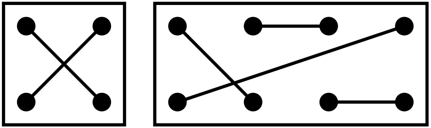

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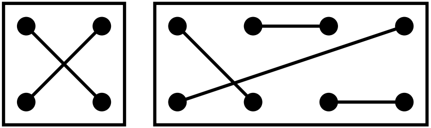

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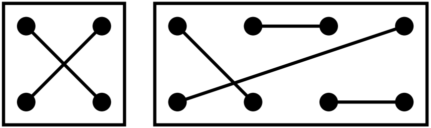

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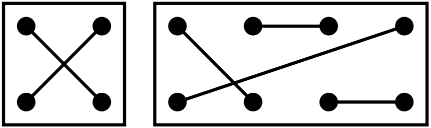

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$D = M_1^G \cup M_2^G$ is (42)-factorisation of index 1

Main results

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Theorem [Bamberg, K., (Schmidt) 2025]

Let $D \subseteq \mathcal{M}_{2n}$ be a non-empty set of perfect matchings and $(a'_\mu)_{\mu \vdash n}$ be its dual distribution. Then

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Corollary [Bamberg, K., (Schmidt) 2025]

Let $\lambda \vdash n$, $\lambda \neq (n)$, and let D be a λ-factorisation of index 1. If k, l with $k \leq l$ are distinct parts of λ, then $2k - 1$ divides $2l + 1$.

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Corollary [Bamberg, K., (Schmidt) 2025]

For $n \geq 4$, a $(n-2, 2)$-factorisation of index 1 can only exist if $n \equiv 0 \mod 3$.

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Cameron: Hyperovals in finite projective planes

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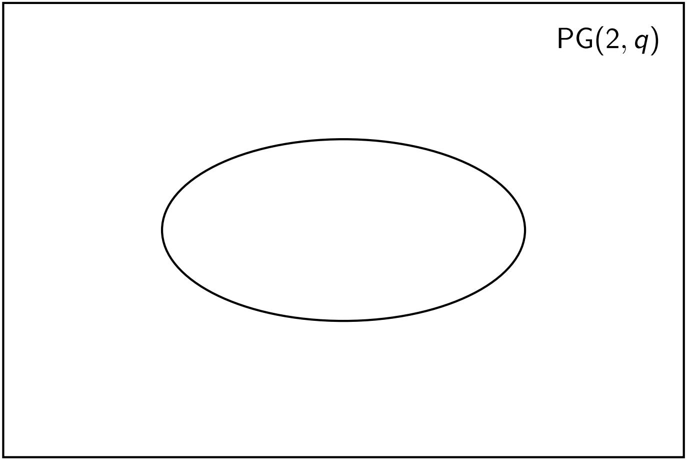

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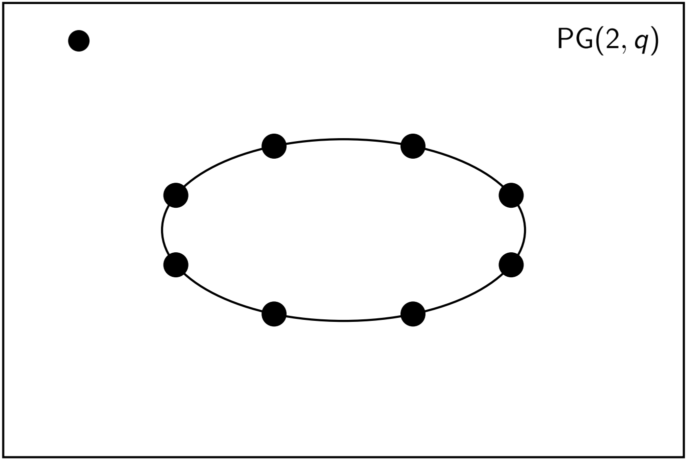

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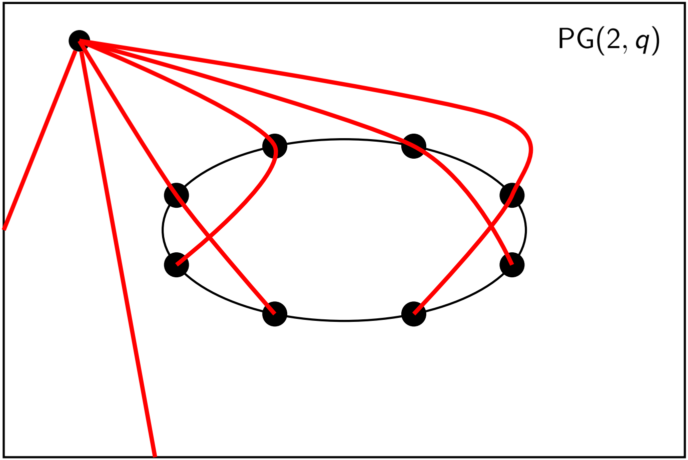

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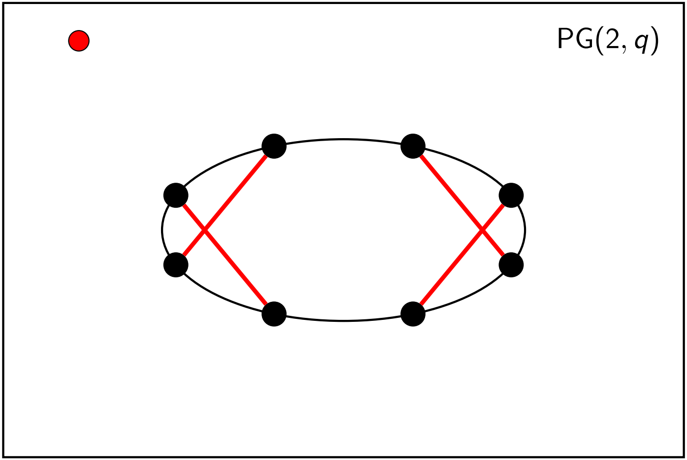

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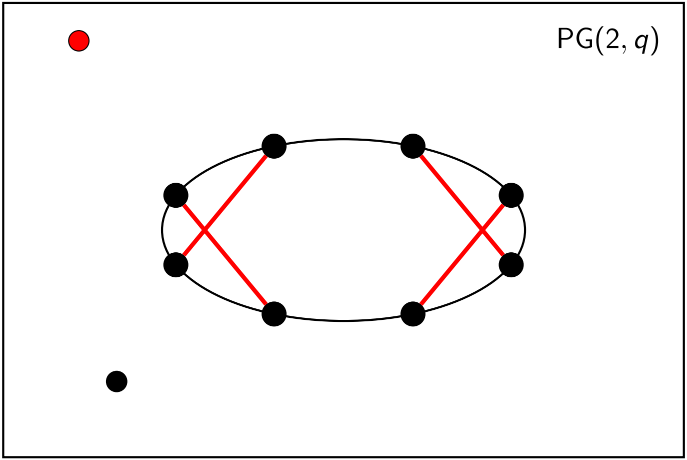

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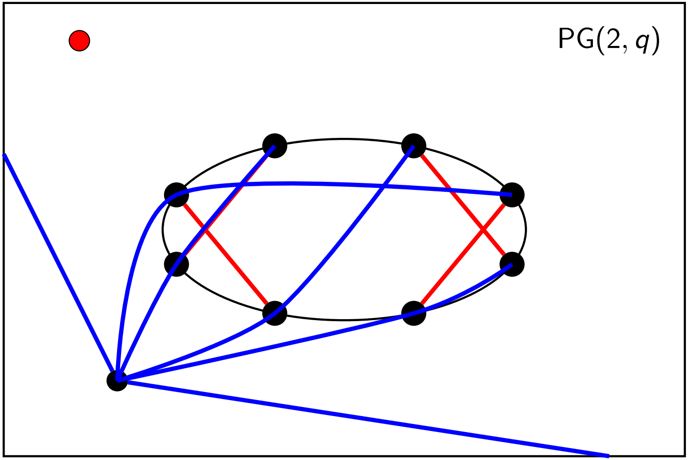

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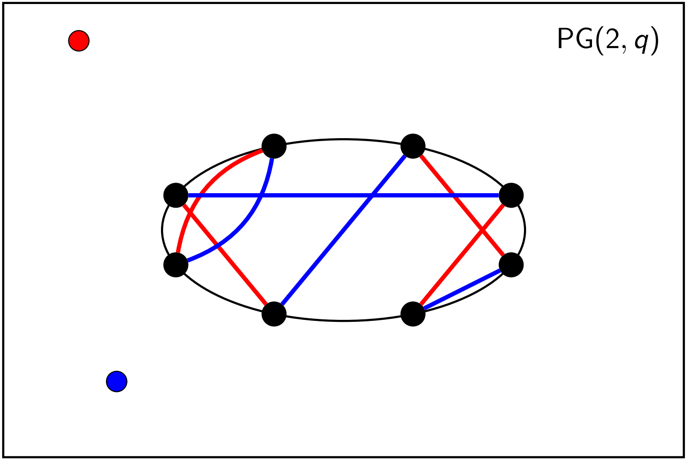

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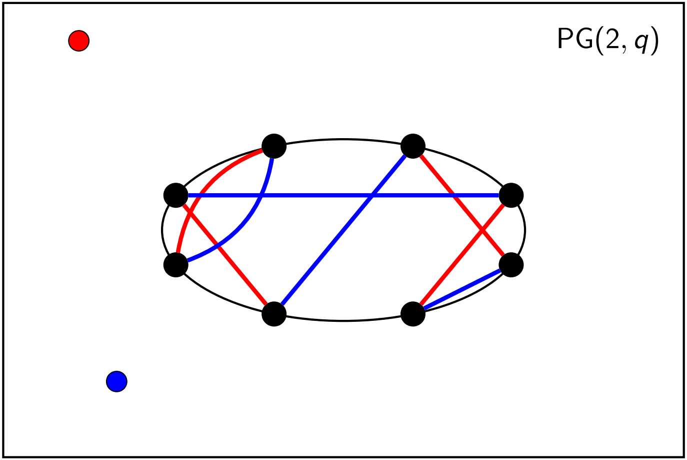

⟶ hyperfactorisation on points of the oval

Thank you for your attention!