Coloring Latin squares by paratopisms

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Example : $\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdot & 4 \\ \hline \cdot & 3 & 4 & \cdot \\ \hline 3 & \cdot & 1 & 2 \\ \hline 4 & 1 & \cdot & \cdot \\ \hline \end{array} \in \mathcal{PL}(4)$

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Definition

Two Latin squares $L_1, L_2 \in \mathcal{L}(n)$ are **isotopic** if there exist a triple $\Theta := (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$ such that $L_1^\Theta = L_2$. Then Θ is called **isotopism**. If $L^\Theta = L$, it is called **autotopism**.

Isotopisms and Parastrophisms in Latin squares.

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$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \xrightarrow{\Theta = ((12),(23),Id_3)} \begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\ \hline 1 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline \end{array}$$

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Two Latin squares $L_1, L_2 \in \mathcal{L}(n)$ are **parastrophic** (commonly known as conjugated) if there exist a permutation $\pi := (\pi_1, \pi_2, \pi_3) \in S_3$ such that $L_1^{\pi} = L_2$. Then π is called **parastrophism**.

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$L \equiv$

1	2	3	4
2	1	4	3
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- We just need to study those autoparatopisms whose structure (Wanless and Mendis, 2016) is one of the following:
  - **Type I**: $((\mathrm{Id}_n, f_2, f_3);(12))$
  - **Type II**: $((\mathrm{Id}_n, \mathrm{Id}_n, f_3);(123))$

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Let $L \in \mathcal{LS}(4)$ and $\sigma \in Par(L)$ as follows:

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$$\sigma = ((Id_4, (12)(34), (12)); (12))$$

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- (1,1,1)

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- (1,1,1) $\rightarrow^{\Theta} (1, 2, 2)$

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$$L \equiv \begin{array}{|c|c|c|c|} \hline \colorbox{red}{1} & 2 & 3 & 4 \\ \hline \colorbox{red}{2} & 1 & 4 & 3 \\ \hline 4 & 3 & 2 & 1 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array} \qquad L^{\Theta} \equiv \begin{array}{|c|c|c|c|} \hline 1 & \colorbox{red}{2} & 4 & 3 \\ \hline 2 & 1 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}$$

- $\color{red}{(1,1,1)}$ $\rightarrow^{\Theta} (1,2,2) \rightarrow^{\pi}$ $\color{red}{(2,1,2)}$

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- $(1,1,1) \rightarrow^{\Theta} (1,2,2) \rightarrow^{\pi} (2,1,2) \rightarrow^{\Theta} (2,2,1)$

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- $(1{,}1{,}1) \to^{\Theta} (1, 2, 2) \to^{\pi} (2{,}1{,}2) \to^{\Theta} (2, 2, 1) \to^{\pi} (2{,}2{,}1)$

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- $(1,1,1) \rightarrow^{\Theta} (1,2,2) \rightarrow^{\pi} (2,1,2) \rightarrow^{\Theta} (2,2,1) \rightarrow^{\pi} (2,2,1) \rightarrow^{\Theta} (2,1,2)$

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- (1,1,1) $\rightarrow^{\Theta} (1,2,2) \rightarrow^{\pi}$ (2,1,2) $\rightarrow^{\Theta} (2,2,1) \rightarrow^{\pi}$ (2,2,1) $\rightarrow^{\Theta} (2,1,2)$ $\rightarrow^{\pi}$ (1,2,2)

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- (1,1,1) $\rightarrow (1,2,2) \rightarrow$ (2,1,2) $\rightarrow (2,2,1) \rightarrow$ (2,2,1) $\rightarrow (2,1,2) \rightarrow$ (1,2,2)
- (1,3,3) $\rightarrow$ ...

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$$\sigma = (\Theta; \pi) = ((Id_4, (12)(34), (12)); (12))$$

$$L \equiv \begin{array}{|c|c|c|c|}\hline \color{red}1 & \color{red}2 & \color{blue}3 & \color{green}4 \\ \hline \color{red}2 & \color{red}1 & \color{green}4 & \color{blue}3 \\ \hline \color{green}4 & \color{blue}3 & \color{pink}1 & \color{pink}2 \\ \hline \color{blue}3 & \color{green}4 & \color{pink}2 & \color{pink}1 \\ \hline \end{array} \qquad L^{\Theta} \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 3 \\ \hline 2 & 1 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}$$

- $\color{red}{(1,1,1)} \to (1,2,2) \to \color{red}{(2,1,2)} \to (2,2,1) \to \color{red}{(2,2,1)} \to (2,1,2) \to \color{red}{(1,2,2)}$
- $\color{blue}{(1,3,3)} \to \ldots$

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Critical sets.

Let $P \in \mathcal{PLS}(5)$:

$$P \equiv \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 5 & \cdot \\ \hline \cdot & 4 & 5 & \cdot & 2 \\ \hline 4 & 5 & \cdot & 2 & 3 \\ \hline \cdot & \cdot & 2 & 3 & 4 \\ \hline \end{array}$$

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·	·	·	·	·
·	·	·	5	·
·	4	5	·	2
4	5	·	2	3
·	·	2	3	4

Let $P \in \mathcal{PLS}(5)$:

$$P \equiv \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 5 & \cdot \\ \hline \cdot & 4 & 5 & \cdot & 2 \\ \hline 4 & 5 & \cdot & 2 & 3 \\ \hline \cdot & \cdot & 2 & 3 & 4 \\ \hline \end{array}$$

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Let $P \in \mathcal{PLS}(5)$:

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·	·	·	·	·
·	·	·	5	·
·	4	5	·	2
4	5	·	2	3
·	·	2	3	4

·	·	·	·	·
·	·	·	5	·
·	4	5	·	2
4	5	·	2	3
·	·	2	3	4

$\xrightarrow{P - forced}$

·	·	·	·	·
·	·	·	5	1
·	4	5	1	2
4	5	1	2	3
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$$\begin{array}{|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 5 & \cdot \\ \hline \cdot & 4 & 5 & \cdot & 2 \\ \hline 4 & 5 & \cdot & 2 & 3 \\ \hline \cdot & \cdot & 2 & 3 & 4 \\ \hline \end{array} \xrightarrow{P-\textit{forced}} \begin{array}{|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 5 & 1 \\ \hline \cdot & 4 & 5 & 1 & 2 \\ \hline 4 & 5 & 1 & 2 & 3 \\ \hline \cdot & \cdot & 2 & 3 & 4 \\ \hline \end{array} \xrightarrow{P-\textit{forced}} \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 & 5 & 1 \\ \hline 3 & 4 & 5 & 1 & 2 \\ \hline 4 & 5 & 1 & 2 & 3 \\ \hline 5 & 1 & 2 & 3 & 4 \\ \hline \end{array}$$

Let $L \in \mathcal{L}(5)$ and $\sigma \in Par(L)$:

$$L \equiv \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 5 & 3 & 2 \\ \hline 2 & 1 & 4 & 5 & 3 \\ \hline 3 & 2 & 1 & 4 & 5 \\ \hline 5 & 3 & 2 & 1 & 4 \\ \hline 4 & 5 & 3 & 2 & 1 \\ \hline \end{array} \qquad \sigma = ((Id_5, (12345), (12)(34)); (12))$$

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σ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\sigma \in Par(L)$:

$$L \equiv \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 5 & 3 & 2 \\ \hline 2 & 1 & 4 & 5 & 3 \\ \hline 3 & 2 & 1 & 4 & 5 \\ \hline 5 & 3 & 2 & 1 & 4 \\ \hline 4 & 5 & 3 & 2 & 1 \\ \hline \end{array} \qquad \sigma = ((Id_5, (12345), (12)(34)); (12))$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 4 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \xrightarrow{\sigma - forced}$$

σ-critical sets.

Let $L \in \mathcal{L}(5)$ and $\sigma \in Par(L)$:

$L \equiv$

1	4	5	3	2
2	1	4	5	3
3	2	1	4	5
5	3	2	1	4
4	5	3	2	1

$$\sigma = ((Id_5, (12345), (12)(34)); (12))$$

1	4	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

1	4	·	3	2
2	1	4	·	3
3	2	1	4	·
·	3	2	1	4
4	·	3	2	1

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$$\sigma = ((Id_5, (12345), (12)(34)); (12))$$

1	4	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

$\xrightarrow{\sigma - forced}$

1	4	·	3	2
2	1	4	·	3
3	2	1	4	·
·	3	2	1	4
4	·	3	2	1

$\xrightarrow[\mathcal{L} - forced]{\sigma - forced}$

σ-critical sets.

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1	4	5	3	2
2	1	4	5	3
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4	5	3	2	1

$$\sigma = ((Id_5, (12345), (12)(34)); (12))$$

1	4	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·
·	·	·	·	·

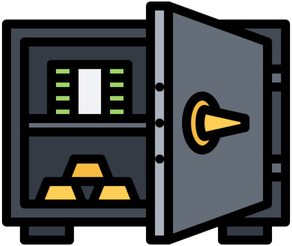
$\xrightarrow{\sigma - forced}$

1	4	·	3	2
2	1	4	·	3
3	2	1	4	·
·	3	2	1	4
4	·	3	2	1

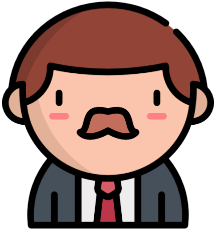
$\xrightarrow[\mathcal{L} - forced]{\sigma - forced}$

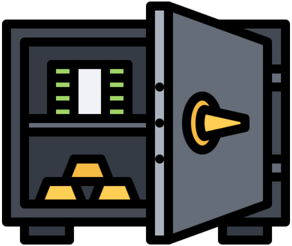
1	4	5	3	2
2	1	4	5	3
3	2	1	4	5
5	3	2	1	4
4	5	3	2	1

Example

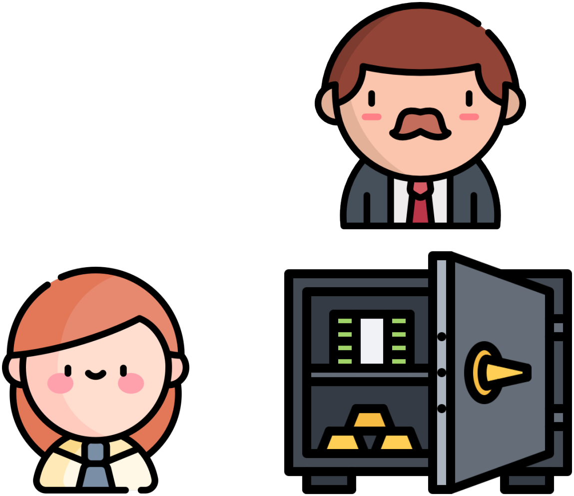


Example

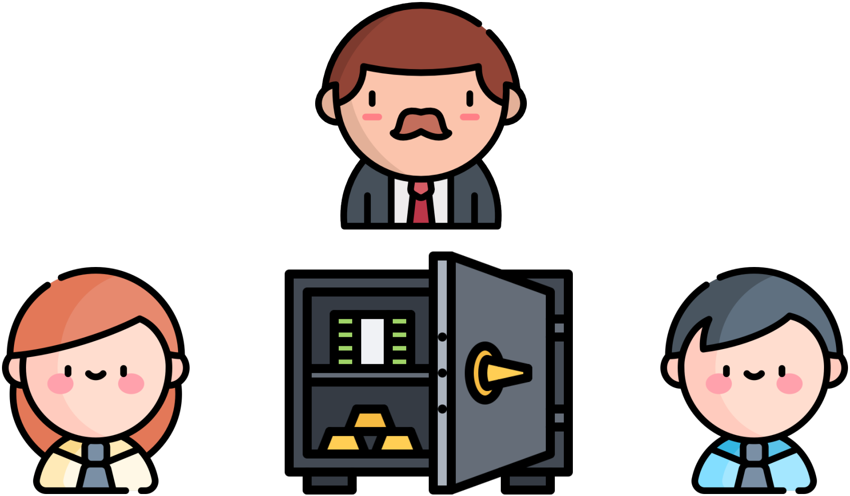




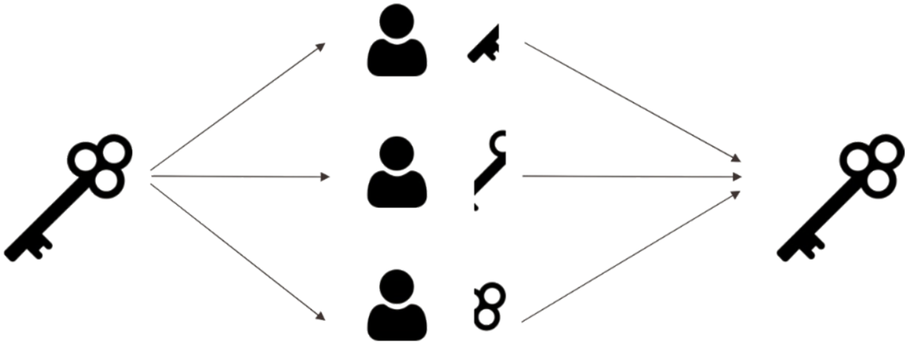
Example



Example



Secret sharing scheme



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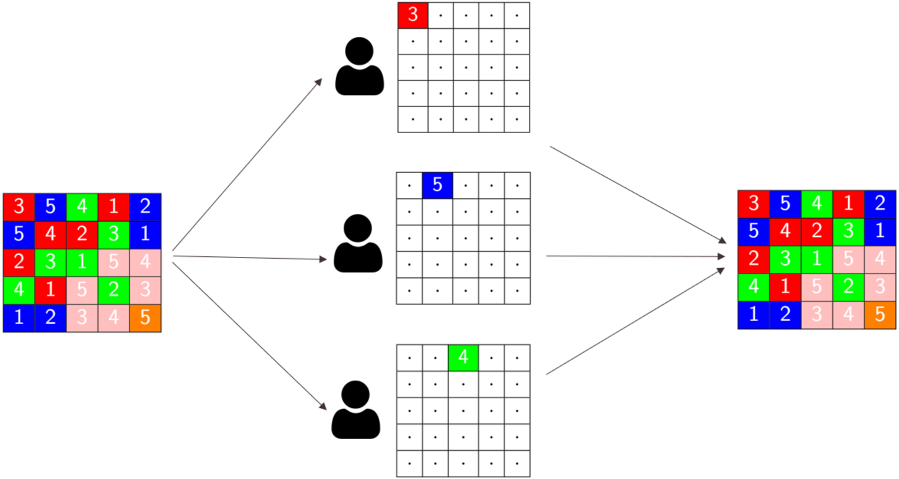

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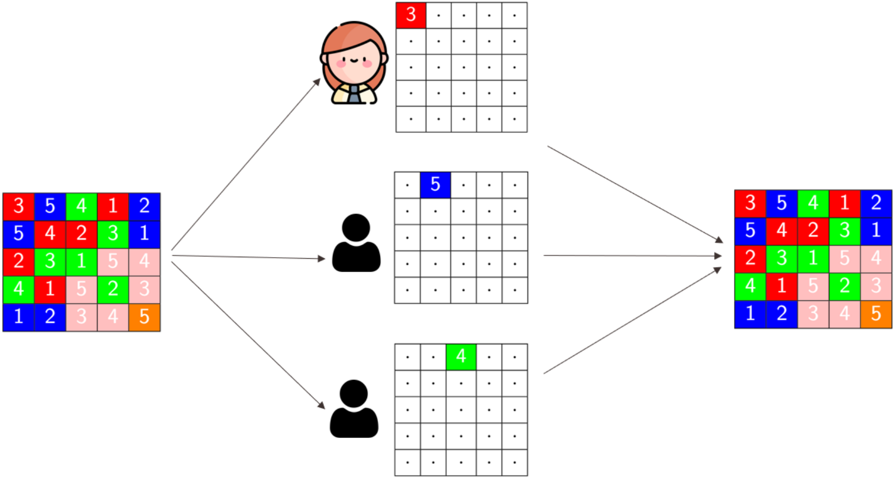

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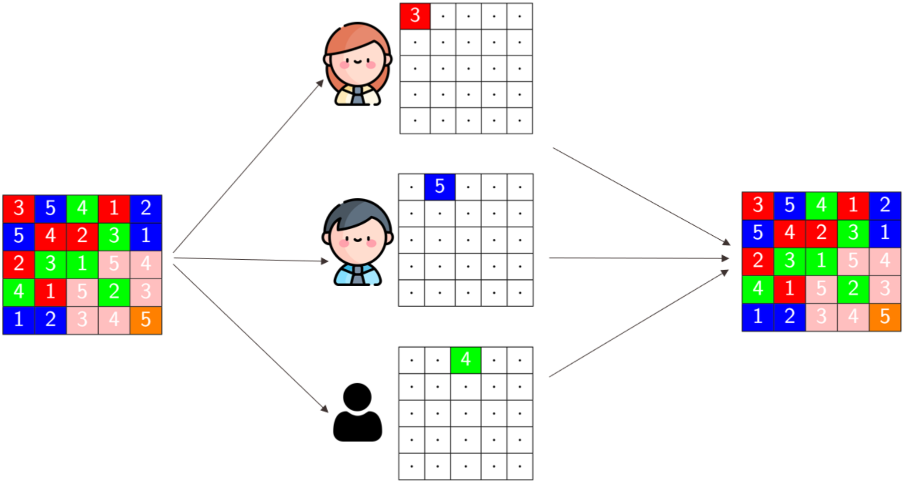

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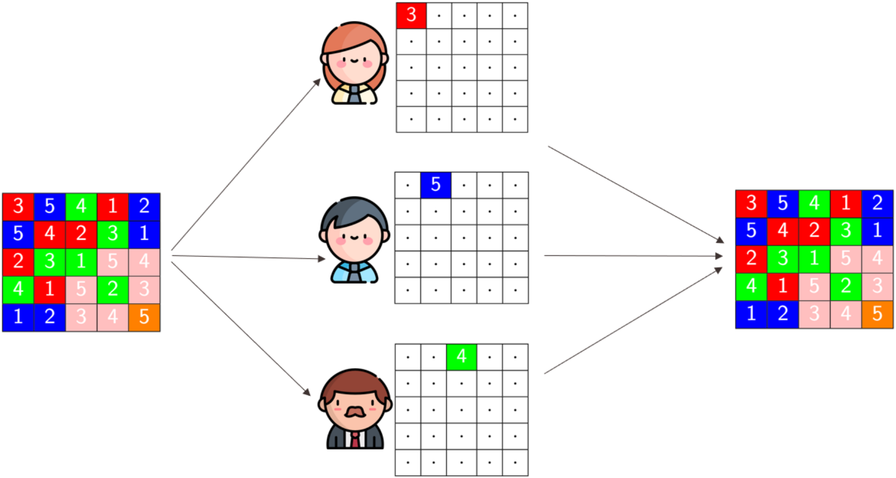

Coloring Latin squares by paratopisms

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