Two-factorizations of some regular graphs

Mariusz Meszka

AGH University of Kraków, Poland

A decomposition of $G = (V,E)$ is a collection $\{H_1, H_2, \ldots, H_t\}$ of edge-disjoint subgraphs of G such that each edge of G belongs to exactly one H_i.

These subgraphs are called blocks of a decomposition.

G has an H-decomposition if all blocks are isomorphic to a given graph H.

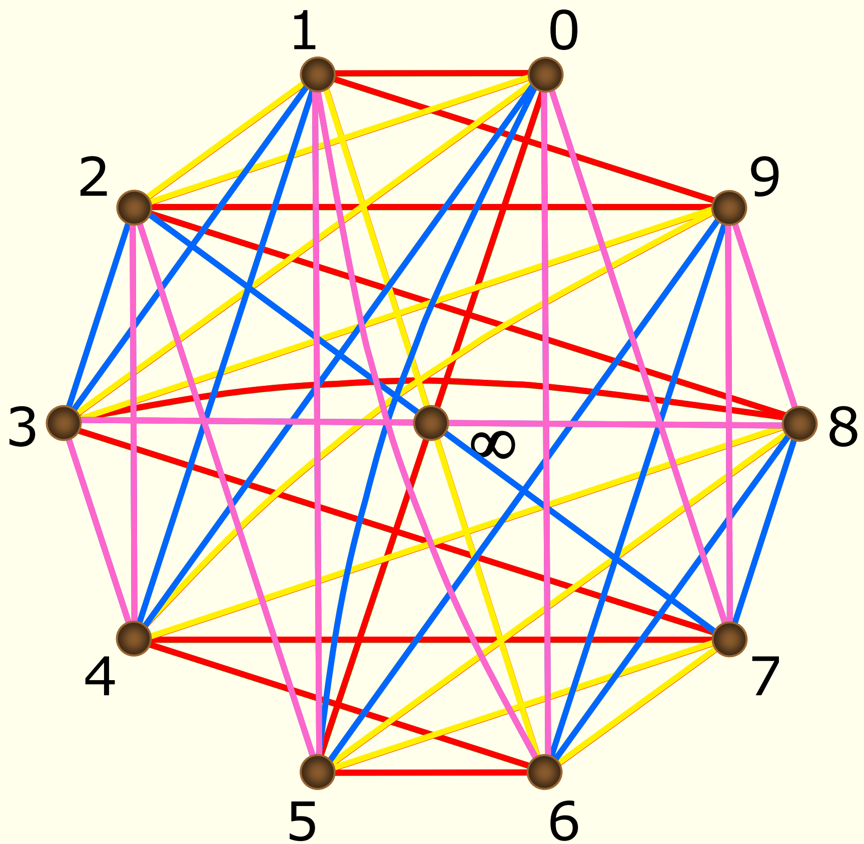
If H is a spanning regular subgraph of degree k in G then H is called a k-factor of G and an H-decomposition is called a k-factorization of G or H-factorization.

Theorem [F. Walecki (1883)]

For each odd $n \geq 3$, the complete graph K_n is decomposable into Hamilton cycles.

Example

$n = 11$

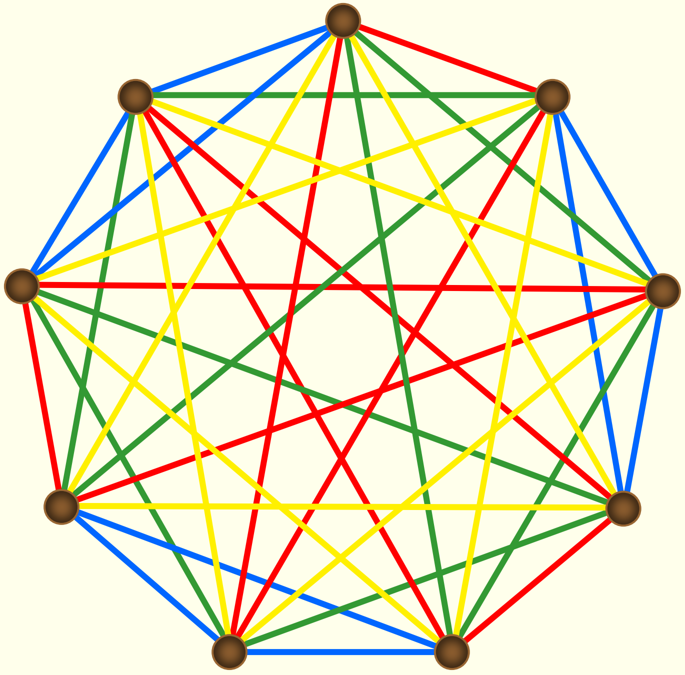


Theorem [D. Ray-Chaudhuri, R. Wilson (1971)]

The complete graph K_n is factorable into C_3-factors if and only if $n \equiv 3 \pmod 6$.

Example

$G=K_9$, $H=C_3$



Theorem [B. Alspach, P. Schellenberg, D. Stinson, D. Wagner (1989)]

For positive integers n and k such that n is odd and $3 \leq k \leq n$, the complete graph K_n has a C_k-factorization if and only if $k \mid n$.

Theorem [A. Kotzig, A. Rosa (1974); R. Baker, R. Wilson (1977); A. Brouwer (1978); R. Rees, D. Stinson (1987)]

The graph $K_n \setminus I$ has a C_3-factorization if and only if $n \equiv 0 \pmod 6$ and $n \geq 18$.

Theorem [B. Alspach, R. Häggkvist (1985); B. Alspach, P. Schellenberg, D. Stinson, D. Wagner (1989); D. Hoffman, P. Schellenberg (1991)]

For any $k \geq 3$, the graph $K_n \setminus I$ has a C_k-factorization if and only if $k \mid n$, except when $k=3$ and $n \in \{6,12\}$.

Oberwolfach Problem [G. Ringel (1967)]

Let $3 \le k_1, k_2, \ldots, k_t$, and $k_1 + k_2 + \ldots + k_t = n$.

Does the complete graph K_n, when n is odd, or $K_n \setminus I$, when n is even, have a 2-factorization in which every 2-factor consists of cycles of lengths $k_1, k_2, \ldots, k_t$?

OP($n; k_1, k_2, \ldots, k_t$) denotes given instance of the Oberwolfach problem.

Exceptions:

- OP(6;3,3)
- OP(9;5,4)
- OP(11;5,3,3)
- OP(12;3,3,3,3)

Theorem [B. Alspach, R. Häggkvist (1985);
D. Bryant, P. Danziger (2012)]

If F is a bipartite 2-factor of order n, then there exists a 2-factorization of K_n - I in which each 2-factor is isomorphic to F.

Theorem [T. Traetta (2013)]

For any $n \geq 7$ and $k \geq 3$, there exists a solution to OP(n;k,n-k), except for OP(9;5,4);

Theorem [B. Alspach, D. Bryant, D. Horsley, B. Maenhaut, V. Scharaschkin (2016)]

If $p \equiv 5 \pmod 8$ is prime, then OP(F) has a solution for every 2-factor F of order $2p$.

An n-vertex graph H is said to be ξ-separable if there exists a set S of at most ξn-vertices such that each component of $H \setminus S$ has size at most ξn.

Theorem [S. Glock, F. Joos, J. Kim, D. Kühn, D. Osthus (2021)]
For given $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exists $\xi_0 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $\xi < \xi_0$.
Let $\mathcal{F}, \mathcal{H}$ be collections of graphs satisfying the following:

(1) $\mathcal{F}$ is a collection of at least αn copies of F, where F is a 2-regular n-vertex graph,

(2) each $H \in \mathcal{H}$ is a ξ-separable n-vertex r_H-regular graph for some $r_H \leq \Delta$,

(3) $e(\mathcal{F} \cup \mathcal{H}) = \binom{n}{2}$.

Then K_n decomposes into $\mathcal{F} \cup \mathcal{H}$.

Computer verification

- $n \le 13$ [F. Franek, J. Holub, A. Rosa (2004)]
- $n \le 17$ [P. Adams, D. Bryant (2006)]
- $n \le 40$ [A. Deza, F. Franek, W. Hua, A. Rosa, MM (2010)]
- $n \le 60$ [F. Salassa, G. Dragotto, T. Traetta, M. Buratti, F. Della Croce (2021)]
- $n \le 100$ [MM (2024)]

Hamilton-Waterloo Problem [A. Rosa (1995)]

Let Q and R be two 2-factors of order n.

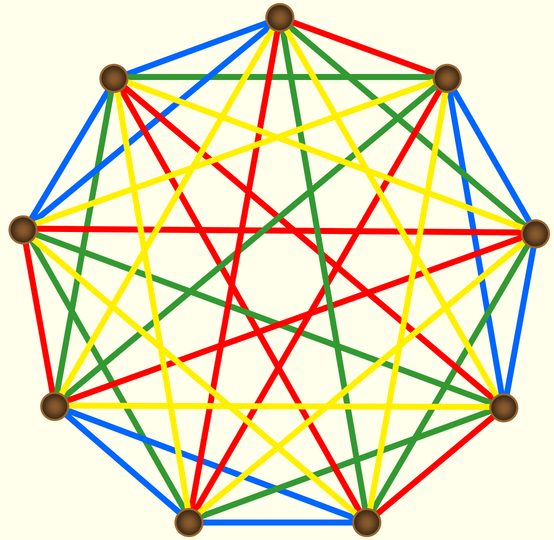
Let q and r be two non-negative integers such that $q + r = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Does the complete graph K_n, when n is odd, or $K_n \setminus I$, when n is even, have a 2-factorization in which
q of the 2-factors are isomorphic to Q and
r of the 2-factors are isomorphic to R?

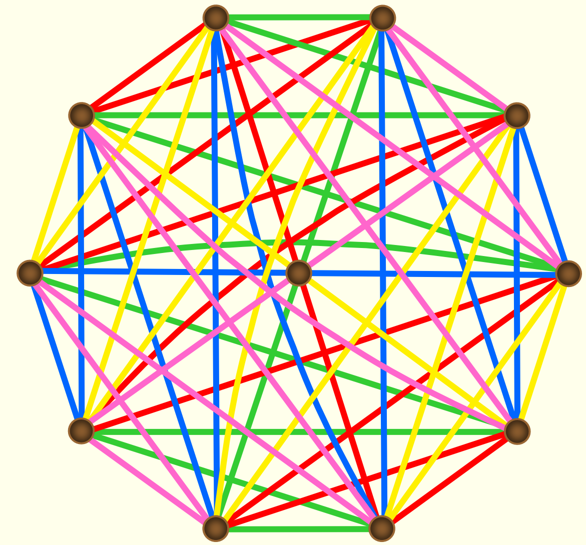
If each component of Q is a cycle of length k and each component of R is a cycle of length l then the corresponding instance of the Hamilton–Waterloo problem is denoted by HW(G;k,l).

Example

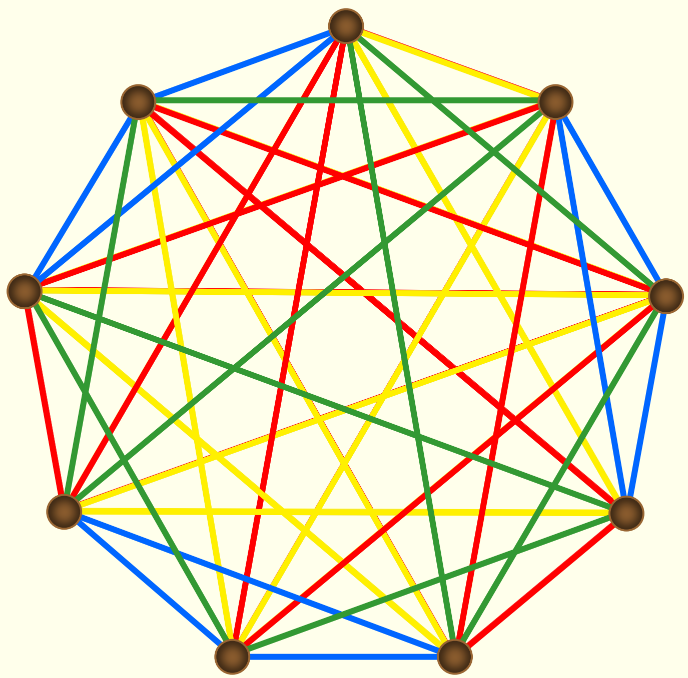
$n=9$
Q: C_3-factor
R: C_n-factor



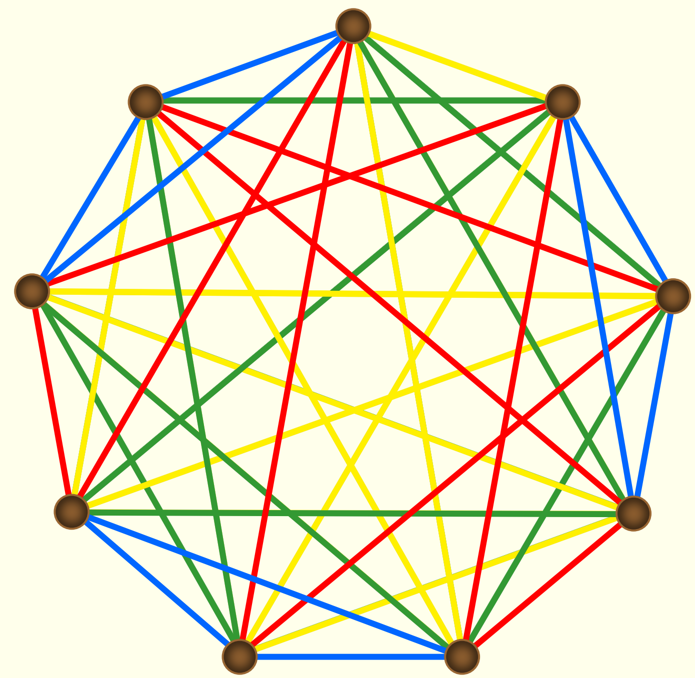
$q=4 \quad r=0$



$q=0 \quad r=4$



$q=2 \quad r=2$



$q=1 \quad r=3$

Theorem [P. Horak, R. Nedela, A. Rosa (2004)]

Let $n \equiv 9 \pmod{18}$. Then $(q,r) \in \mathrm{HW}(K_n;3,n)$ if and only if $q, r \geq 0$, $q+r = \frac{n-1}{2}$, and $r \neq 1$ when $n=9$.

Theorem [P. Horak, R. Nedela, A. Rosa (2004)]

Let $n \equiv 3$ or $15 \pmod{18}$. Assume that

$r \in \{\frac{n+3}{6}, \frac{n+3}{6}+2, \frac{n+3}{6}+3, \ldots, \frac{n-1}{2}\}$ and $q = \frac{n-1}{2} - r$.

Then $(q,r) \in \mathrm{HW}(K_n;3,n)$.

Theorem [A. Ling, J. Dinitz (2009)]

Let $n \equiv 3 \pmod{18}$ and $n \geq 57$. Then $(q,r) \in \mathrm{HW}(K_n;3,n)$ if and only if $q,\ r \geq 0$, $q+r = \frac{n-1}{2}$, except possibly when $r=1$.

Theorem [J. Dinitz, A. Ling (2009)]

For each $n \equiv 3 \pmod{6}$, $(\frac{n-3}{2},1) \in \mathrm{HW}(K_n;3,n)$ except when $n=9$ and with the possible exceptions of

$n \in \{93,111,123,129,141,153,159,177,183,201,207,213,$
$237,249\}$.

Theorem [MM (2025+)]

For each $n \equiv 3 \pmod{6}$, $(q,r) \in \mathrm{HW}(K_n;3,n)$ if and only if $q,\ r \geq 0$, $q+r = \frac{n-1}{2}$, except when $n=9$ and $r=1$.

Theorem [A. Burgess, P. Danziger, T. Traetta (2018)]

Let m,v,q,r be integers such that m and v are odd, $m \geq v \geq 3$ and q, $r > 0$. Then $(q,r) \in \mathrm{HW}(K_{mv};m,mv)$ if and only if $q+r = \frac{mv-1}{2}$, except possibly when at least one of the following holds:

(1) $r=1$

(2) $q < \frac{m-1}{2}$

(3) $q - \frac{m-1}{2} \in \{1,3\}$ and $m > v$

(4) $(m,v)=(5,3)$ and $q - \frac{m-1}{2} \equiv 1,3 \pmod{m}$

Theorem [S. Özkan, MM (2025+)]

Let m,v be odd integers such that $m \geq 7$ and $v \geq 3$. Then $(q,r) \in \mathrm{HW}(K_{mv};m,mv)$ if and only if $q+r = \frac{mv-1}{2}$ with q, $r \geq 0$.

Theorem [S. Özkan, MM (2025+)]

Let $v \geq 3$ be an odd integer. Then $(q,r) \in \mathrm{HW}(K_{5v};5,5v)$ if and only if $q+r = \frac{5v-1}{2}$ with $q,\ r \geq 0$, except possibly when $1 \leq r \leq \frac{v-1}{2}$ for $v \geq 7$ and $5 \nmid v$.

Theorem [A. Burgess, P. Danziger, T. Traetta (2018)]

Let m,v,p,r,s be integers such that m and v are odd, $m,v \geq 3$ and $q,r,s > 0$. Then $(q,r) \in \mathrm{HW}(K_{smv};m,mv)$ if and only if $q+r=\frac{smv-1}{2}$, except possibly when at least one of the following holds:

(1) $r=1$

(2) $(m,q)=(3,6)$

(3) $s \in \{1,2,4\}$ and either $v > m$ or one of the following holds:

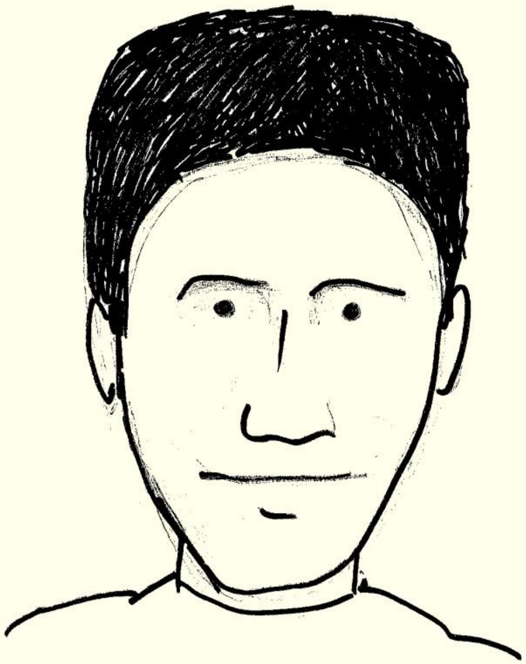
- (a) $q < \left\lfloor\frac{sm-1}{2}\right\rfloor$
- (b) $q - \left\lfloor\frac{sm-1}{2}\right\rfloor \in \{1,3\}$ and $m > v$
- (c) $(m,v)=(5,3)$ and $q - \left\lfloor\frac{sm-1}{2}\right\rfloor \equiv 1,3 \pmod{m}$
- (d) $(v,s)=(3,2)$

Theorem [S. Özkan, MM (2025+)]

Let m,v,s be integers such m,v are odd, $m \geq 7$, $v \geq 3$, $s \geq 1$. Then $(q,r) \in \mathrm{HW}(K_{smv};m,mv)$ if and only if $q+r = \left\lfloor \frac{smv-1}{2} \right\rfloor$ with q, $r \geq 0$.

Tools and constructions

- graph products: Cartesian, lexicographic, direct, strong
- circulants
- Skolem-type sequences
- cycle designs
- block colorings of designs
- efficient computer search



Thank you!