Point-line geometries related to binary equidistant codes

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The language of points and lines

A *point-line geometry* $(\mathcal{L}, \mathcal{P})$ is a pair, where $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines:

- each line contains at least three points,
- the intersection of two distinct lines contains at most one point.

Distinct points are *collinear* if there is a line containing them.

The *collinearity graph* of $(\mathcal{L}, \mathcal{P})$ is the simple graph whose vertices are points and two points are connected by an edge if they are collinear.

A *subspace* is a subset $\mathcal{S} \subset \mathcal{P}$ such that for any collinear points $x, y \in \mathcal{S}$ the line joining x, y is contained in $\mathcal{S}$.
A subspace is *singular* if any two distinct points of this subspace are collinear.

Example: Polar space

Let Ω be a non-degenerate reflexive sesquilinear form on a vector space V.
A non-zero vector $x \in V$ is *isotropic* if $\Omega(x, x) = 0$.
A subspace $S \subset V$ is *totally isotropic* if $\Omega(x, y) = 0$ for any $x, y \in S$.

The associated polar space:
The points are 1-dimensional totally isotropic subspaces.
The lines correspond to 2-dimensional totally isotropic subspaces.
The maximal singular subspaces correspond to maximal totally isotropic subspaces. They are of the same dimension which is called the *Witt index* of Ω.

Our plans

We consider a point-line geometry whose maximal singular subspaces correspond to an equivalence class of equidistant codes.

A large portion of material concerns the case of binary equidistant codes, binary simplex codes and their applications to symmetric block designs.

At the end, some words on geometries of non-binary simplex codes.

Binary equidistant codes

Let $\mathbb{F}$ be the field of two elements and $\mathbb{F}^n$ be an n-dimensional vector space over this field.
A *binary linear* $[n, k]$ *code* is a k-dimensional subspace of $\mathbb{F}^n$.

The *Hamming weight* of $v \in \mathbb{F}^n$ is the number of non-zero coordinates of v.
The *Hamming distance* between $v, w \in \mathbb{F}^n$ is the Hamming weight of $v - w$.

A code $C \subset \mathbb{F}^n$ is t-*equidistant* if the Hamming distance between any distinct codewords $c, c' \in C$ is t, equivalently, all non-zero codewords in C are of Hamming weight t.

Example: The binary $[7, 3]$ code with the generator matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

is 4-equidistant.
This code also is a *binary simplex code of dimension* 3.

Binary simplex codes

Suppose that $n = 2^k - 1$.
Note that $2^k - 1$ is the number of 1-dimensional subspaces of $\mathbb{F}^k$, i.e. the number of points in the projective space $PG(k-1, 2)$.

A k-dimensional code $C \subset \mathbb{F}^n$ is a binary *simplex* code if in every generator matrix of C all columns are non-zero and mutually distinct, i.e. there is a one-to-one correspondence between the columns and points of $PG(k-1, 2)$.

Every binary simplex code of dimension k is 2^{k-1}-equidistant.
Every two such codes are *equivalent*, i.e. there is a monomial linear automorphism of $\mathbb{F}^n$ transferring one of these codes to the other.

Theorem 1 (A. Bonisoli)**.** *Every equidistant code is equivalent to a code obtained by replication of a simplex code and adding some zero coordinates to each code word.*

Some examples of binary equidistant codes

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

is the generator matrix of the binary 8-equidistant $[14, 3]$ code which is the 2-replication of a 3-dimensional binary simplex code.

To obtain more equidistant codes we can permute columns

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and add zero columns

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The projective space $PG(n-1, 2)$. Set theoretical approach

As above, $\mathbb{F}$ is the field of two elements and $\mathbb{F}^n$ is a vector space over this field.

Every point $P = \langle x_1, \ldots, x_n \rangle$, $x_i \in \{0, 1\}$ of the associated projective space $PG(n-1, 2)$ can be identified with the non-empty set $I \subset [n] = \{1, \ldots, n\}$ formed by all indices $i \in [n]$ such that $x_i \neq 0$.
Then every line of $PG(n-1, 2)$ is a triple of subsets $I, J, I \triangle J$, where $I \triangle J$ is the symmetric difference.

We will consider $PG(n-1, 2)$ as the set of all non-empty subsets of $[n]$ whose lines are triples $I, J, I \triangle J$.
The Hamming weight of the point corresponding to $I \subset [n]$ is $|I|$ and the Hamming distance between I, J is $|I \triangle J|$.

Subgeometries of points with fixed Hamming weight

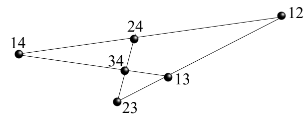
Let m be an integer satisfying $1 \le m \le n/2$.
Denote by $\mathcal{P}_m(n)$ the set of all subsets $I \subset [n]$ such that $|I| = 2m$, i.e. the set of all points of $PG(n-1,2)$ with Hamming weight $2m$.
Consider $\mathcal{P}_m(n)$ as a point-line geometry whose lines are the lines of $PG(n-1,2)$ contained in $\mathcal{P}_m(n)$.
Then $I, J \in \mathcal{P}_m(n)$ are collinear if and only if $|I \triangle J| = m$ or, equivalently, $|I \cap J| = m$.

We assume that $n \ge 3m$.
Reason: If $n < 3m$, then for any subsets $I, J \subseteq [n]$ satisfying $|I| = |J| = 2m$ we have $|I \cap J| > m$ which implies that $|I \triangle J| < 2m$, i.e. such $\mathcal{P}_m(n)$ does not contain lines.
For the same reason, every set formed by all points of fixed odd Hamming weight does not contain lines (if $I, J \subseteq [n]$ and $|I| = |J|$ is odd, then $|I \triangle J|$ is even).

Singular subspaces of the geometry $\mathcal{P}_m(n)$ are subspaces of $PG(n-1,2)$ corresponding to $2m$-equidistant codes.
Maximal singular subspaces are of the same dimension, the corresponding equidistant codes form an equivalence class.
One important case: If $n = 2^{k-1} - 1$ and $m = 2^{k-2}$, then maximal singular subspaces of $\mathcal{P}_m(n)$ correspond to binary simplex codes of dimension k.

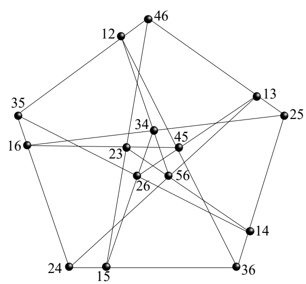
Examples

1. $\mathcal{P}_1(4)$ is the Pasch configuration.



2. $\mathcal{P}_2(6)$ is the Cremona-Richmond configuration.

Every $P_I \in \mathcal{P}_2$ is identified with the 2-element subset $[6] \setminus I$ and three points of $\mathcal{P}_2$ form a line if and only if the corresponding 2-element subsets are mutually disjoint.



3. $\mathcal{P}_2(7)$ is a polar space.
All vectors $(x_1, \ldots, x_7) \in \mathbb{F}^7$ of Hamming weight 4 form the quadric

$$\sum_{i=1}^{7} x_i = 0 \text{ and } \sum_{i<j} x_i x_j = 0.$$

Binary simplex codes of dimension 4 are maximal singular subspace of this quadric.

Automorphism of $\mathcal{P}_m(n)$

An *automorphism* of the geometry $\mathcal{P}_m(n)$ is a bijective transformation preserving the family of lines in both directions.
Every permutation on $[n]$ induces an automorphism of $\mathcal{P}_m(n)$.

In some cases, there are automorphisms of $\mathcal{P}_m(n)$ which are not induced by permutations.

Example 1. Suppose that $n = 4m - 1$ and fix $i \in [n]$. Consider $f_i : \mathcal{P}_m(n) \to \mathcal{P}_m(n)$

$$f_i(I) = I \ \text{ if } \ i \notin I \ \text{ and } \ f_i(I) = ([n] \setminus I) \cup \{i\} \ \text{ if } \ i \in I.$$

Example 2. Suppose that $n = 4m$ and fix distinct $i, j \in [n]$. Consider $f_{ij} : \mathcal{P}_m(n) \to \mathcal{P}_m(n)$

$$f_{ij}(I) = I \ \text{ if } \ i, j \in I \text{ or } i, j \notin I \ \text{ and } \ f_{ij}(I) = [n] \setminus I \ \text{ if } I \text{ contains precisely one of } i, j.$$

Theorem 2 (M.P., K. Petelczyc, M. Żynel)**.** *Every automorphism of the geometry $\mathcal{P}_m(n)$ is induced by a permutation on $[n]$ or is the composition of an automorphism induced by a permutation and one of the automorphisms considered in Examples 1, 2.*

More details at Mariusz Żynel talk (Thursday, June 5, 17:10-17:30).

Maximal cliques of the collinearity graph and symmetric block designs

Let $\Gamma_m(n)$ be the collinearity graph of the geometry $\mathcal{P}_m(n)$, i.e. the simple graph whose vertices are points of $\mathcal{P}_m(n)$ and two vertices are connected by an edge if they are collinear points.

Every maximal singular subspace of $\mathcal{P}_m(n)$ is a maximal clique of $\Gamma_m(n)$.

- Maximal cliques of $\Gamma_m(3m)$ are precisely lines of $\mathcal{P}_m(3m)$.
- $\mathcal{P}_2(7)$ is a polar space which implies that maximal cliques of $\Gamma_2(7)$ coincide maximal singular subspaces of $\mathcal{P}_2(7)$.

In the remaining cases, $\Gamma_m(n)$ contains maximal cliques which are not maximal singular subspaces of $\mathcal{P}_m(n)$.

By Fisher's inequality, every maximal clique of $\Gamma_m(n)$ contains no more than n elements.
Every n-clique of $\Gamma_m(n)$ is formed by blocks of a certain symmetric $(n, 2m, m)$-design.

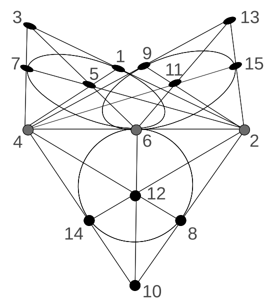
Geometric description of the five symmetric $(15, 8, 4)$-designs

It is well-known that there are precisely five symmetric $(15, 8, 4)$-designs.

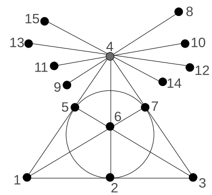
M.P., K. Petelczyc, M. Żynel obtained the following geometric description of these designs as maximal cliques of $\Gamma_4(15)$.

(1) Maximal singular subspace isomorphic to $PG(3, 2)$.

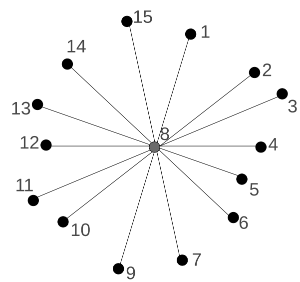
(2) Three Fano planes through a line.



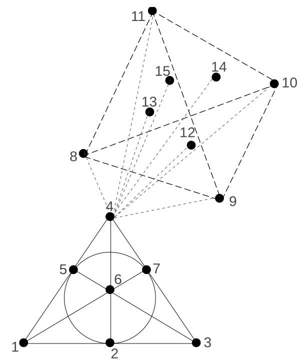
(3) Fano plane and four lines through a point



(4) Eight lines through a point



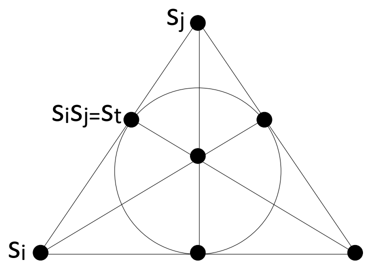
(5) $PG(3,2)$ minus Fano plane plus a disjoint Fano plane.



More details at Krzysztof Petelczyc's talk (Thursday, June 5, 16:50-17:10).

Automorphisms of $(15, 8, 4)$-designs

Fano plane can be considered as the group $\mathcal{F}$ consisted of the identity e and involutions $s_1, \ldots, s_7$ corresponding to the points of Fano plane.
The product is $s_i s_j = s_t$, where s_t corresponds to the third points on the line joining the points related to s_i and s_j.



The automorphism group of a symmetric $(15, 8, 4)$-design of type (1) (i.e. a maximal singular subspace isomorphic to $PG(3, 2)$) is $GL(4, 2)$.

Theorem 3 (M.P., K. Petelczyc, M. Żynel). *The automorphism group of a symmetric* $(15, 8, 4)$*-design of type (2)-(5) is the semidirect product of the group* $\mathcal{F}$ *and a subgroup of* $GL(3, 2)$*. This subgroup is dependent on the design type.*

Geometries of non-binary simplex codes

A *q-ary simplex code of dimension* k is a non-degenerate linear $[\frac{q^k-1}{q-1}, k]_q$ with the property that the columns in every generator matrix are mutually non-proportional, i.e. there is a one-to-one correspondence between the columns and points of the projective space $PG(k-1, q)$.

Consider the point-line geometry $\mathcal{S}(k, q)$ whose points and lines are 1-dimensional and 2-dimensional subcodes of q-ary simplex codes of dimension k and maximal singular subspaces correspond to q-ary simplex code of dimension k.

Every maximal clique of the collinearity graph of $\mathcal{S}(k, q)$ contains no more than $n = \frac{q^k-1}{q-1}$ vertices.
Every maximal singular subspace of $\mathcal{S}(k, q)$ is a maximal clique of the collinearity graph.
For $q \geq 5$, M. Kwiatkowski, M. P., A. Tyc constructed a class of n-cliques distinct from maximal singular subspaces.
If $k = 2$ (maximal singular subspaces are lines), then some of such cliques are projectively equivalent to the normal rational curve

$$\{\langle 1, t, \ldots, t^{m-1} \rangle : t \in \mathbb{F}_q\} \cup \{\langle (0, \ldots, 0, 1) \rangle\},$$

in other words, they are arcs.

More details at Adam Tyc's talk (Thursday, June 5, 9:50-10:10).