Robinson–Schensted Shapes Arising from Cycle Decompositions

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5th Pythagorean Conference, June 2025

A Tale of Two Partitions

Much classical algebraic combinatorics $\rightsquigarrow$
representation theory of the *symmetric group* S_n $\rightsquigarrow$
integer partitions

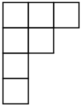
For $n, \lambda_i \in \mathbb{N}$, we say that

$$\lambda = (\lambda_1, \dots, \lambda_r)$$

is a **partition** of n, written

$$\lambda \vdash n,$$

if λ is weakly decreasing and $\sum_{i=1}^{r} \lambda_i = n$. Example: $(3, 2, 1, 1) \vdash 7$.

$$(3, 5, 4, 7)(1, 2, 6) \xrightarrow{?}$$ 

The First Partition: Cycle Type

Two notations for $\sigma \in S_n$:

- *one-line notation*: $\sigma = [2, 4, 9, 7, 10, 3, 1, 14, 6, 12, 11, 5, 13, 8]$
- *cycle notation:* $\sigma = (1, 2, 4, 7)(3, 9, 6)(5, 10, 12)(8, 14)(11)(13)$.

The **cycle type** of σ is the partition of n given by the lengths of the disjoint cycles in the cycle notation for σ.

In the example above, σ has cycle type $(4, 3, 3, 2, 1, 1)$.

For $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, let

$$\mathcal{C}_\alpha := \{\sigma \in S_n : \sigma \text{ has cycle type } \alpha\}.$$

Definition of Young Diagram

The **Young diagram** of shape $\lambda \vdash n$ is a left-aligned array of cells with λ_i boxes in the ith row, counting from the top.

As an example, the Young diagram of shape $(5, 5, 4, 2, 1, 1)$ is:

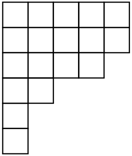


Figure: Young diagram of shape $(5, 5, 4, 2, 1, 1)$

Definition of Standard Young Tableau

A **standard Young tableau** of shape $\lambda \vdash n$ is obtained from the Young diagram of λ by

1. filling the cells with distinct elements of $[1, n]$
2. so that entries in each row and column are strictly increasing

We write $\mathrm{SYT}(n)$ for the set of standard Young tableaux of size n.

1	2	3	4	5
6	7	8	9	18
10	11	12	17	
13	16			
14				
15				

Figure: A standard Young tableau of shape $(5, 5, 4, 2, 1, 1)$

Robinson–Schensted Correspondence

The **Robinson–Schensted (RS) correspondence** is a bijection

$$S_n \xrightarrow{\mathrm{RS}} \coprod_{\lambda \vdash n} \mathsf{SYT}(\lambda) \times \mathsf{SYT}(\lambda)$$

written $\mathsf{RS}(\sigma) = (P, Q)$.

Shorthand Algorithm. Let $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$, and let $\mathsf{RS}(\sigma) = (P, Q)$. Inductively:

- For P: try to add σ_k to the end the first row
- For P: otherwise put it in its proper position, bumping out what was there already and try to place the bumped element in the next row...
- For Q: record where the new box was added in P with a k

Robinson–Schensted Example

$$P: \begin{array}{c}2\end{array} \to \begin{array}{cc}2 & 6\end{array} \to \begin{array}{cc}2 & 5\\ 6 & \end{array} \to \begin{array}{ccc}2 & 5 & 7\\ 6 & & \end{array} \to \begin{array}{ccc}2 & 4 & 7\\ 5 & & \\ 6 & & \end{array}$$

$$\to \begin{array}{ccc}1 & 4 & 7\\ 2 & & \\ 5 & & \\ 6 & & \end{array} \to P = \begin{array}{ccc}1 & 3 & 7\\ 2 & 4 & \\ 5 & & \\ 6 & & \end{array}$$

$$Q: \begin{array}{c}1\end{array} \to \begin{array}{cc}1 & 2\end{array} \to \begin{array}{cc}1 & 2\\ 3 & \end{array} \to \begin{array}{ccc}1 & 2 & 4\\ 3 & & \end{array} \to \begin{array}{ccc}1 & 2 & 4\\ 3 & & \\ 5 & & \end{array}$$

$$\to \begin{array}{ccc}1 & 2 & 4\\ 3 & & \\ 5 & & \\ 6 & & \end{array} \to Q = \begin{array}{ccc}1 & 2 & 4\\ 3 & 7 & \\ 5 & & \\ 6 & & \end{array}$$

Figure: Given the input $\sigma = [2, 6, 5, 7, 4, 1, 3] = (3, 5, 4, 7)(1, 2, 6)$, the RS algorithm produces the output $\mathrm{RS}(\sigma) = (P, Q)$.

The Second Partition: RS Shape

For $\sigma \in S_n$, write $\mathrm{RS}(\sigma) = (P, Q)$. Note that P and Q share the same shape; if this shape is $\lambda \vdash n$, then we call λ the **RS shape** of σ, and we write

$$\mathrm{sh}(\sigma) = \lambda.$$

From the last example, $\sigma = [2, 6, 5, 7, 4, 1, 3] = (3, 5, 4, 7)(1, 2, 6)$, we had

$$\mathrm{RS}(\sigma) = \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \right)$$

so

$$\mathrm{sh}(\sigma) = (3, 2, 1, 1).$$

Main Goal

Given a cycle type

$$\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$$

and having defined the conjugacy classes $\mathcal{C}_\alpha$ in S_n, along with the RS shape of an element $\sigma \in S_n$,

$$\lambda = \mathrm{sh}(\sigma) \vdash n,$$

we now introduce our main object of study.

Define

$$\mathcal{S}_\alpha := \{\mathrm{sh}(\sigma) : \sigma \in \mathcal{C}_\alpha\}.$$

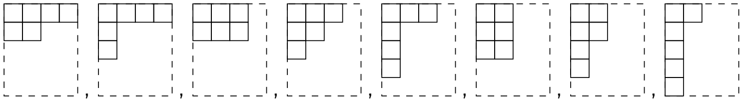
Our main result describes $\mathcal{S}_\alpha$ for all cycle types $\alpha = (\alpha_1, \alpha_2)$.

The Bounding Box

For $\alpha = (\alpha_1, \ldots, \alpha_r)$, define

$$\mathcal{B}_\alpha := \left\{ \lambda \vdash n : \begin{array}{l} \lambda_1' \leq n - r + \#\{i : \alpha_i = 2\} + \delta_{1,\alpha_r}, \\ \lambda_1 \leq n - r + \#\{i : \alpha_i = 1\} \end{array} \right\}.$$

λ_1' denotes the number of rows of λ and λ_1 denotes the number of columuns.
For example, if $\alpha = (4, 2)$, then $r = 2$ and $\mathcal{B}_\alpha$ consists of all partitions of $n = 6$ whose Young diagram fits inside the box of dimensions $(6 - 2 + 1 + 0) \times (6 - 2 + 0) = 5 \times 4$. Concretely, we have

$\mathcal{B}_{(4,2)} = \Bigg\{$  $\Bigg\}.$

Global Constraint

Theorem ($\mathcal{S}_\alpha$ Box Constraint)

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$*. Then* $\mathcal{S}_\alpha \subseteq \mathcal{B}_\alpha$.

Idea of proof:
A result of Schensted (generalized by Greene) shows that the number of columns (resp., rows) in $\lambda = \mathrm{sh}(\sigma)$ equals the length of the longest ascending (resp., descending) subsequence in $[\sigma_1, \ldots, \sigma_n]$. With this tool, analysis of maximal ascending (resp., descending) subsequences in cycles gives the result.

Main Result

Theorem (Identification of $\mathcal{S}_\alpha$ for $r = 2$)

Let $n \in \mathbb{N}$, and let $\alpha = (\alpha_1, \alpha_2) \vdash n$.

1. *If n is odd, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$.*
2. *If n is even, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ unless α occurs in the following table:*

α	$\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha$
$(n-1,\ 1)$	$\{(\frac{n}{2}, \frac{n}{2})\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \mid n$	$\{(n-2,1,1),\ (3,1,\ldots,1)\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \nmid n$	$\{(n-2,1,1)\}$
$(4,2)$	$\{(2,2,2)\}$
$(5,3)$	$\{(2,2,2,2)\}$

Admissible Tableaux

$T_\lambda :=$ the tableau in $\mathrm{SYT}(\lambda)$ with column word $[1, \ldots, n]$.

$$T_{(3,2,1,1)} = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

Given a standard Young tableau Q, we define

$Q^{\uparrow} :=$ the tableau obtained from Q by reversing the entries in each column.

$$T_{(3,2,1,1)} = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \qquad T^{\uparrow}_{(3,2,1,1)} = \begin{array}{|c|c|c|}\hline 4 & 6 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}.$$

Definition (Admissible Tableaux)

A standard tableau Q is said to be *admissible* if the entries in $Q^{\uparrow}$ increase from left to right along each row.

Raison d'Etre for Admissible

The group S_n acts naturally on a tableau T of size n by permuting entries.

Corollary (Main Use of Admissible)

Let $\lambda \vdash n$ and let $Q \in \mathrm{SYT}(\lambda)$ be admissible. If you find

$$\sigma \in S_n,$$

define

$$P = \sigma \cdot Q^{\uparrow},$$

and can verify that

$$P \in \mathrm{SYT}(\lambda)$$

then $\mathrm{RS}(\sigma) = (P, Q)$, and we have

$$\mathrm{sh}(\sigma) = \lambda.$$

In particular, in the special case $Q = T_\lambda$, the one-line notation of σ is the column word of $P^{\uparrow}$.

Warm-Up, $r = 1$, and the Canonical Cycle

Here is the solution to the main problem for the baby case, $r = 1$:

$$\mathcal{S}_{(n)} = \mathcal{B}_{(n)}.$$

Construction (Canonical Cycle–Rough Outline)

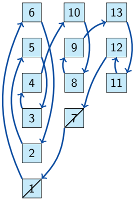
Let $\lambda \in \mathcal{B}_{(n)}$. Construct a canonical σ as follows:

1. *Begin with T_λ, which is admissible.*
2. *Draw spiral arrows in each column of $T_\lambda^\uparrow$ starting in the first column and working to the right.*
3. *After the last column, work your way back to the first column along the bottom.*
4. *The arrows give σ^{-1}. Hence to write σ in cycle notation, simply follow the arrows backwards.*

Picture of the Canonical Cycle for $\lambda = (3, 3, 3, 2, 1, 1)$

6	10	13
5	9	12
4	8	11
3	7	
2		
1		

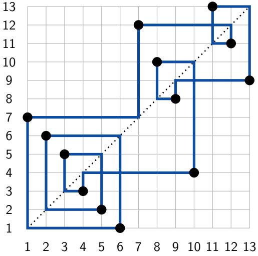
(a) $T^{\uparrow}_{\lambda}$



(b) Arrows giving σ^{-1}

1	4	9
2	8	11
3	10	13
5	12	
6		
7		

(c) $P = \sigma \cdot T^{\uparrow}_{\lambda}$



(d) Graph of σ

Arrows $\rightarrow$ cycle notation; reverse column word of P $\rightarrow$ one-line notation:

$$\sigma^{-1} = (1, 6, 2, 5, 3, 4, 10, 8, 9, 13, 11, 12, 7)$$
$$\sigma = [7, 6, 5, 3, 2, 1, 12, 10, 8, 4, 13, 11, 9].$$

Via the RS correspondence, we have $\text{RS}(\sigma) = (P, T_{\lambda})$.

Beyond $r = 1$, α-Colorings

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, $\lambda \in \mathcal{B}_\alpha$, and $Q \in \mathrm{SYT}(\lambda)$ admissible. An *α-coloring of $Q^\uparrow$* is a partitioning of the boxes of $Q^\uparrow$ into the colors $c_1, \ldots, c_r$, such that

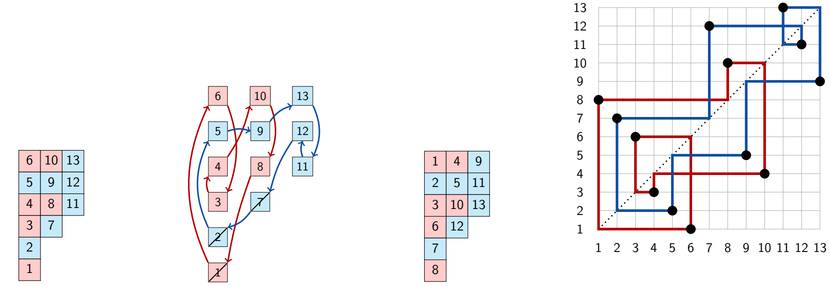
1. there are exactly α_i boxes with color c_i
2. upon cyclically permuting the entries of each color via the arrows from the canonical constructions, while ignoring the boxes of other colors, one obtains a standard tableau $P \in \mathrm{SYT}(\lambda)$.

The *associated permutation* of an α-coloring is the permutation σ such that $P = \sigma \cdot Q^\uparrow$.

Theorem

Let $\alpha, \lambda \vdash n$ and let $Q \in \mathrm{SYT}(\lambda)$ be admissible. If there exists an α-coloring of $Q^\uparrow$, then $\lambda \in \mathcal{S}_\alpha$.

α-Coloring Picture for $\alpha = (7, 6)$ and $\lambda = (3, 3, 3, 2, 1, 1)$



(a) $Q^{\uparrow} = T^{\uparrow}_{\lambda}$ (b) Arrows giving σ^{-1} (c) $P = \sigma \cdot Q^{\uparrow}$ (d) Graph of σ

Arrows $\to$ cycle notation; reverse column word of P $\to$ one-line notation:

$$\sigma^{-1} = (1, 6, 3, 4, 10, 8)(2, 5, 9, 13, 11, 12, 7)$$
$$\sigma = [8, 7, 6, 3, 2, 1, 12, 10, 5, 4, 13, 11, 9].$$

Via the RS correspondence, we have $\mathrm{RS}(\sigma) = (P, T_\lambda)$.

Idea for Proof of Main Theorem

The main theorem was for $\alpha = (\alpha_1, \alpha_2) \vdash n$ and said that $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ with 5 exceptions for even n.

Idea of proof: Construct α-colorings that apply to every shape $\lambda \in \mathcal{B}_\alpha$ except for the table of exceptions.

Strict Partition Conjecture

We say that $\alpha \vdash n$ is a *strict partition* if $\alpha_1 > \cdots > \alpha_r$. The following conjecture is verified by computer for all $n \leq 15$.

Conjecture

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$ be a strict partition, where $r \geq 3$.

1. *If $\alpha_r > 1$, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$.*
2. *If $\alpha_r = 1$ and n is odd, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$.*
3. *If $\alpha_r = 1$ and n is even, then $\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha = \{(\frac{n}{2}, \frac{n}{2})\}$.*

The following conjecture is verified by computer for all $n \leq 13$.

Conjecture

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$ be a strict partition, where $r \geq 1$. For each $\lambda \in \mathcal{S}_\alpha$, there exists an admissible $Q \in \mathrm{SYT}(\lambda)$ such that $Q^\uparrow$ admits an α-coloring.

Further Problems

Problem

For general cycle types $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, describe $\mathcal{S}_\alpha$ explicitly.

Answer known in one special case (Schützenberger): Let $\alpha = (2^{r-k},\ 1^k)$. Then

$$\mathcal{S}_\alpha = \{\lambda \vdash n : \lambda \text{ has exactly } k \text{ many columns of odd length}\}.$$

Further Problem

Conjecture ("Almost" Pieri rule)

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, where $\alpha_r > 1$ and $r \geq 1$. Fix a positive integer k, and set

$$\widetilde{\alpha} \coloneqq (\alpha_1, \ldots, \alpha_r, 1^k).$$

Then we have

$$S_{\widetilde{\alpha}} = \left\{ \lambda \in \mathcal{B}_{\widetilde{\alpha}} : \begin{array}{l} \text{Young diagram of } \lambda \text{ is obtained from the Young diagram} \\ \text{of some } \mu \in \mathcal{S}_\alpha \text{ by adding exactly } k \text{ boxes, with no two} \\ \text{in the same column, and such that if } \lambda_1' = 2 \text{ then all } k \\ \text{boxes must be added to the first row} \end{array} \right\}.$$

($\lambda_1' = 2$ means λ has two rows)

The End

Thank you!