Pursuit-evasion on graphs arising from combinatorial designs

Nancy E. Clarke
Acadia University

Joint work with:
Andrea Burgess (UNBSJ)
Rosalind Cameron (Canterbury)
Peter Danziger (TMU)
Stephen Finbow (StFX)
Caleb Jones (MUN)
David Pike (MUN)

Rules of the Game

- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately
- cops win if at least one of them occupies the same vertex as the robber after a finite number of moves
- the copnumber $c(G)$ is the minimum number of cops that suffice to guarantee a win on G
- a graph with copnumber 1 is copwin

Rules of the Surrounding Cops game

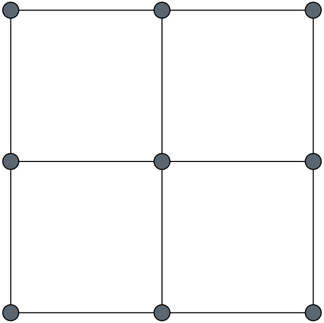
- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately

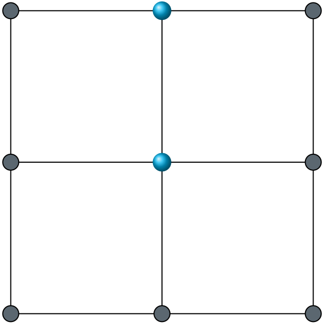
Rules of the Surrounding Cops game

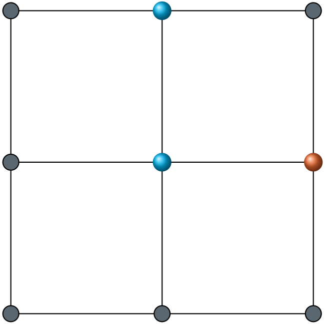
- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately
- the robber cannot start on or move to a vertex occupied by a cop
- the cops win if, at any time, each of the neighbours of the robber's vertex is occupied by a cop

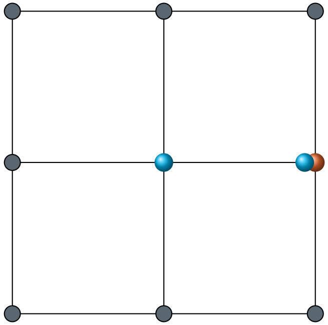
Rules of the Surrounding Cops game

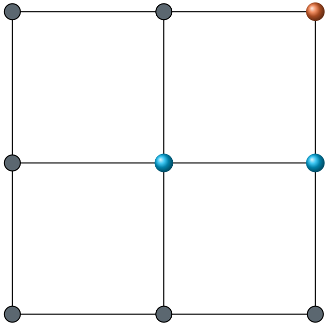
- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately
- the robber cannot start on or move to a vertex occupied by a cop
- the cops win if, at any time, each of the neighbours of the robber's vertex is occupied by a cop
- the surrounding copnumber $\sigma(G)$ is the minimum number of cops that suffice to guarantee a win

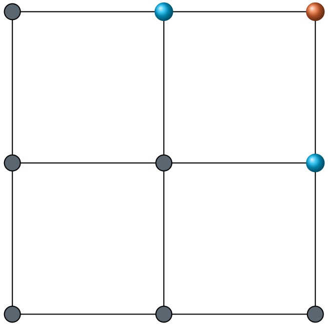






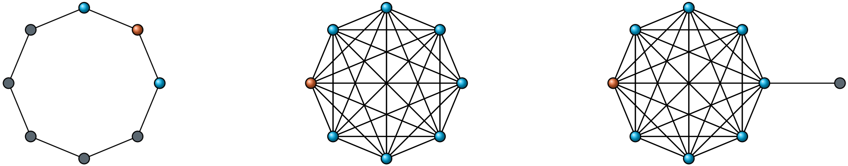






Lower bounds on $\sigma(G)$

Lemma For any graph G, $\sigma(G) \geq \min\{\delta(G), c(G), \omega(G) - 1\}$.



$c(C_n) = 2$	$c(K_n) = 1$	$c(K_n + e) = 1$
$\delta(C_n) = 2$	$\delta(K_n) = n - 1$	$\delta(K_n + e) = 1$
$\omega(C_n) = 2$	$\omega(K_n) = n$	$\omega(K_n + e) = n$
$\sigma(C_n) = 2$	$\sigma(K_n) = n - 1$	$\sigma(K_n + e) = n - 1$

More Bounds

Theorem If G has girth $g \geq 7$ and minimum degree $\delta \geq 3$, then $\sigma(G) \geq \delta + 1$.

More Bounds

Theorem If G has girth $g \geq 7$ and minimum degree $\delta \geq 3$, then $\sigma(G) \geq \delta + 1$.

Theorem If H is a retract of G, then $\sigma(H) \leq \sigma(G)$.

More Bounds

Theorem If G has girth $g \geq 7$ and minimum degree $\delta \geq 3$, then $\sigma(G) \geq \delta + 1$.

Theorem If H is a retract of G, then $\sigma(H) \leq \sigma(G)$.

Lemma For any graph G, $\sigma(G) \leq |V(G)| - \alpha(G)$.

Corollary For any graph G, $\alpha(G) + \omega(G) \leq |V(G)| + 1$.

More Bounds

Theorem If G has girth $g \geq 7$ and minimum degree $\delta \geq 3$, then $\sigma(G) \geq \delta + 1$.

Theorem If H is a retract of G, then $\sigma(H) \leq \sigma(G)$.

Lemma For any graph G, $\sigma(G) \leq |V(G)| - \alpha(G)$.

Corollary For any graph G, $\alpha(G) + \omega(G) \leq |V(G)| + 1$.

Theorem For any graph G, $\sigma(G) \leq tw(G) + 1$.

Corollary If G is a chordal graph then $\sigma(G) \in \{\omega(G) - 1, \omega(G)\}$.

Can we characterize those graphs G with $\sigma(G) = n$?

- $\sigma(G) = 1$: the only graphs are K_1 and $K_{1,n}$

Can we characterize those graphs G with $\sigma(G) = n$?

- $\sigma(G) = 1$: the only graphs are K_1 and $K_{1,n}$
- $\sigma(G) = 2$: paths of length at least 3, cycles, cycles with a chord, most trees

Can we characterize those graphs G with $\sigma(G) = n$?

- $\sigma(G) = 1$: the only graphs are K_1 and $K_{1,n}$
- $\sigma(G) = 2$: paths of length at least 3, cycles, cycles with a chord, most trees
- $\sigma(G) = 3$: wheels

Graph products

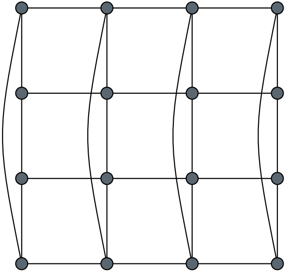
We define 3 graph products in terms of edge matrices:
$$\begin{array}{c} \\ E \\ \Delta \\ N \end{array}\begin{array}{c} \begin{array}{ccc} E & \Delta & N \end{array} \\ \begin{pmatrix} - & - & - \\ - & \Delta & - \\ - & - & - \end{pmatrix} \end{array}.$$

Cartesian: $G \,\square\, H \quad \begin{pmatrix} N & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix}$; Strong: $G \boxtimes H \quad \begin{pmatrix} E & E & N \\ E & \Delta & N \\ N & N & N \end{pmatrix}$

Lexicographic: $G \bullet H \quad \begin{pmatrix} E & E & E \\ E & \Delta & N \\ N & N & N \end{pmatrix}$

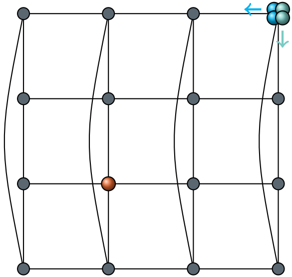
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



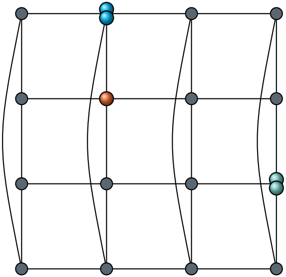
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



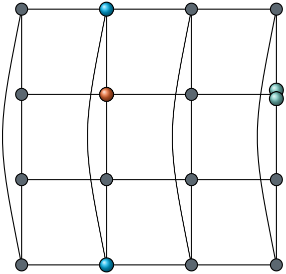
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



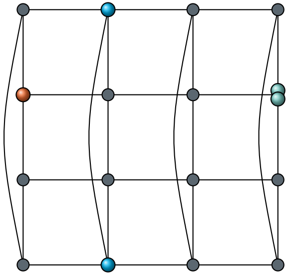
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



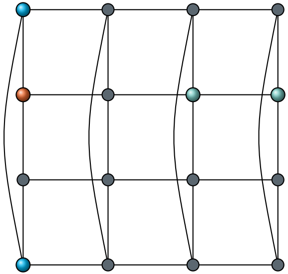
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



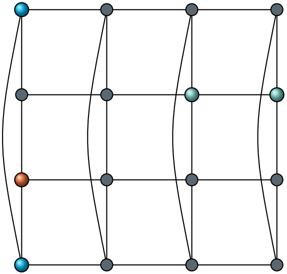
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \Box H) \leq \sigma(G) + \sigma(H)$.



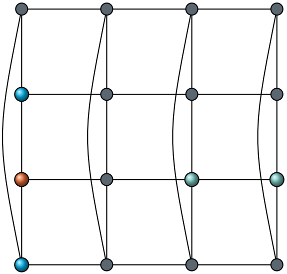
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



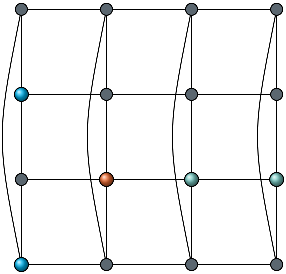
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \Box H) \leq \sigma(G) + \sigma(H)$.



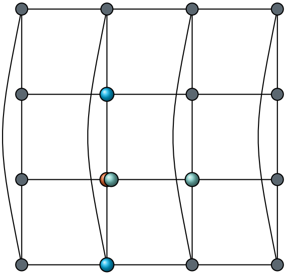
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \Box H) \leq \sigma(G) + \sigma(H)$.



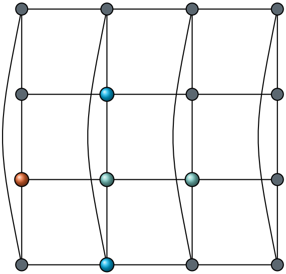
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \Box H) \leq \sigma(G) + \sigma(H)$.



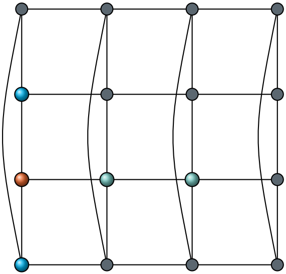
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



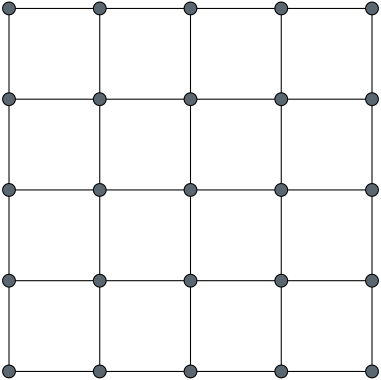
Cartesian product

Theorem For connected graphs G and H, $\sigma(G \square H) \leq \sigma(G) + \sigma(H)$.



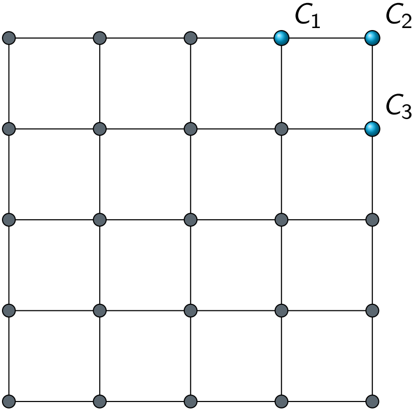
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



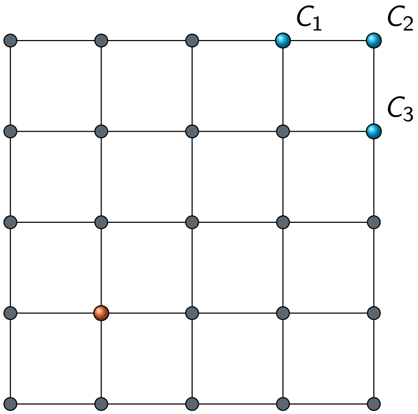
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



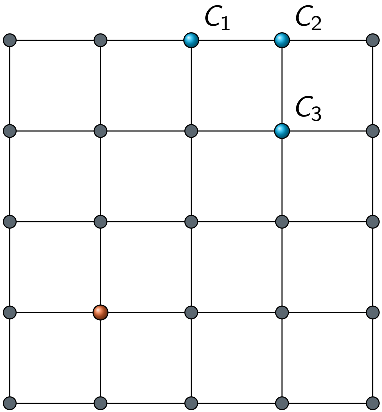
Cartesian products of paths

Theorem Let $2 \le m \le n$ be integers. If $m, n \le 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.

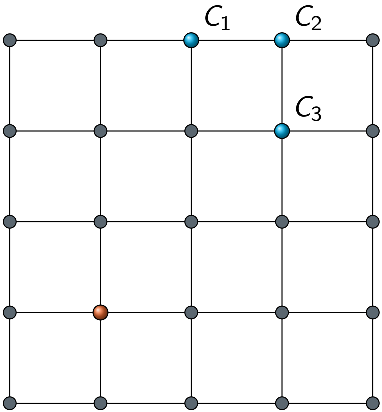


Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.

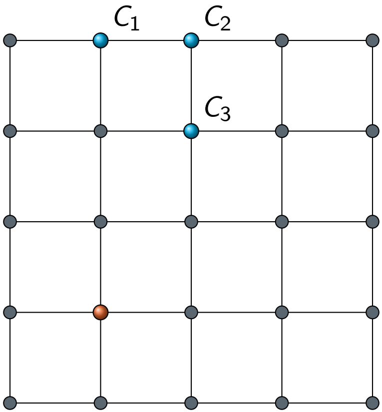


Theorem Let $2 \le m \le n$ be integers. If $m, n \le 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



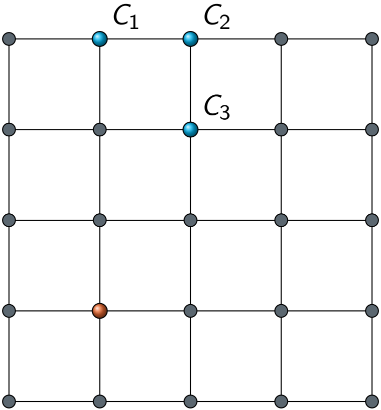
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



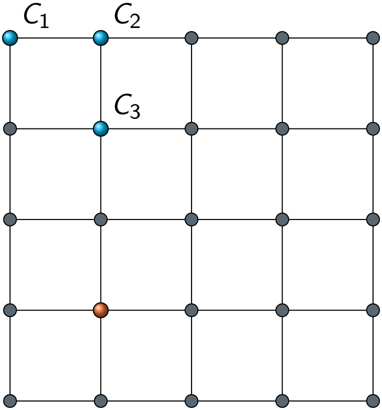
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



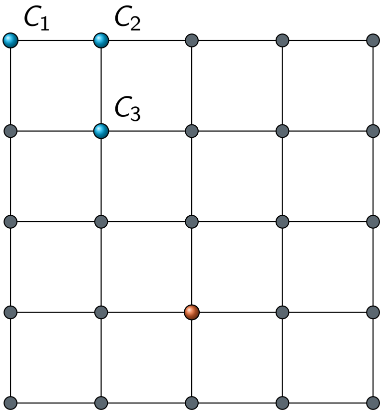
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



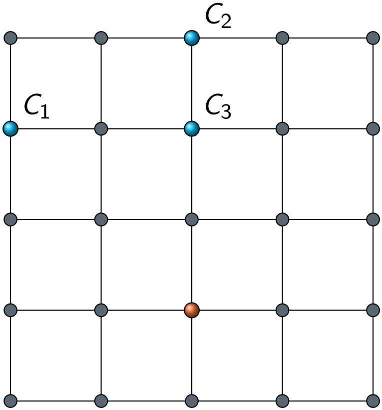
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



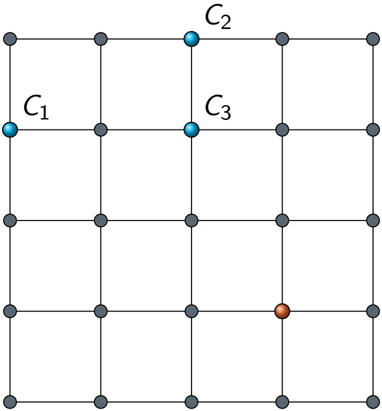
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



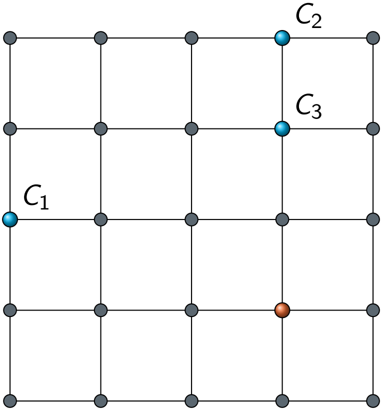
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



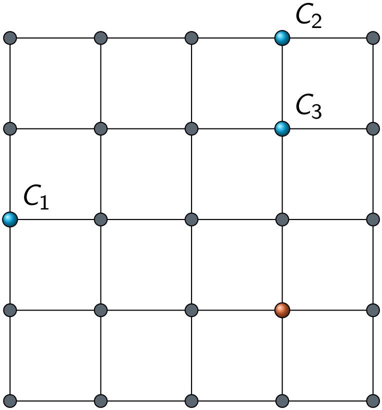
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



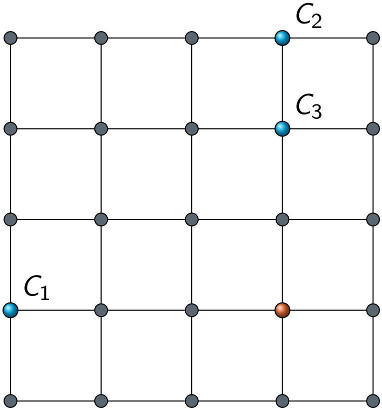
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



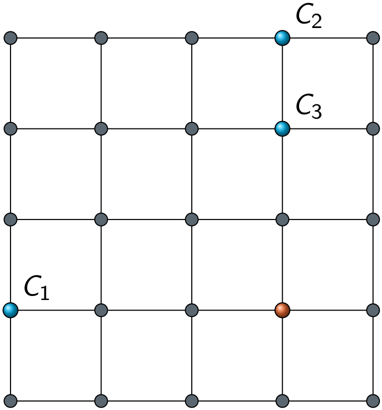
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



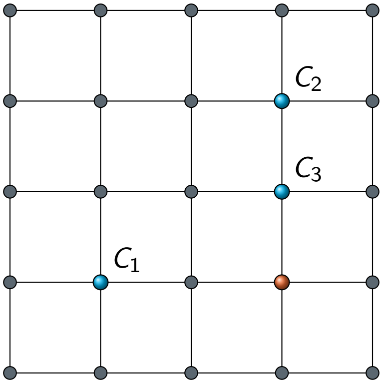
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



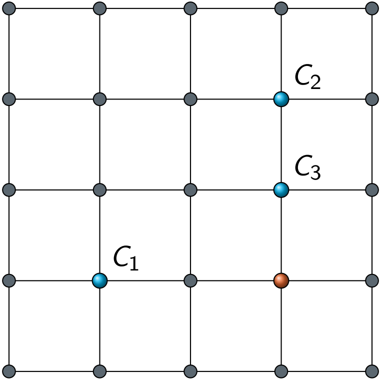
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



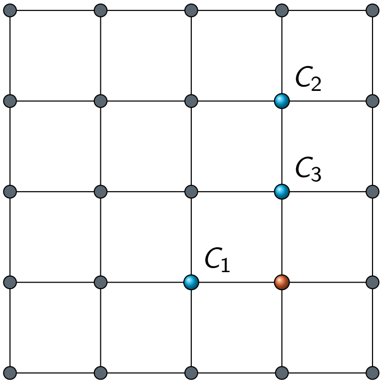
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



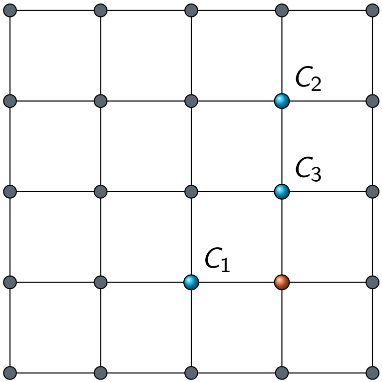
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



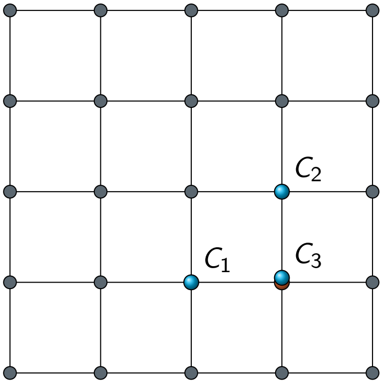
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



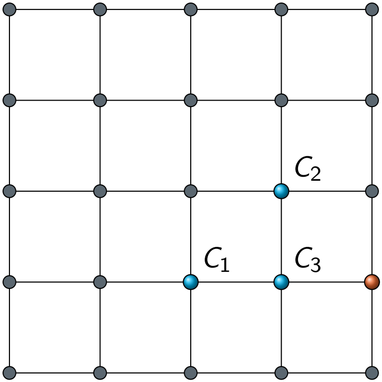
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



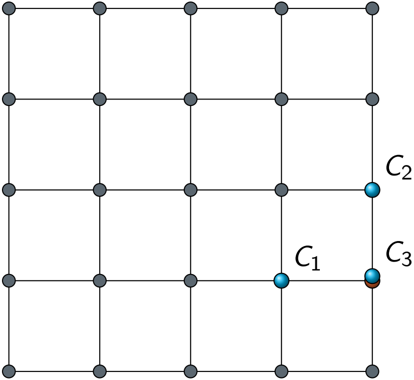
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



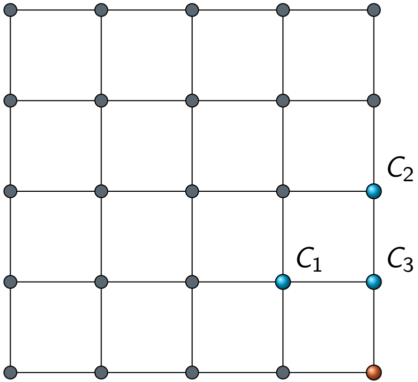
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \Box P_n) = 2$; otherwise $\sigma(P_m \Box P_n) = 3$.



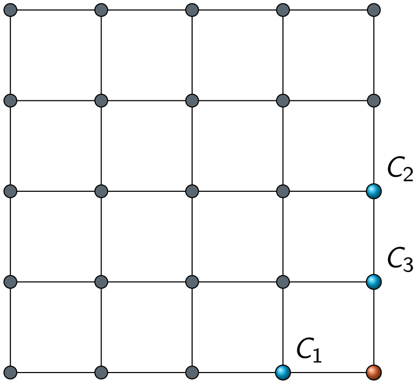
Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



Cartesian products of paths

Theorem Let $2 \leq m \leq n$ be integers. If $m, n \leq 3$ then $\sigma(P_m \square P_n) = 2$; otherwise $\sigma(P_m \square P_n) = 3$.



Strong Products

Theorem Let $2 \leq m \leq n$ be integers. Then

$$\sigma(P_m \boxtimes P_n) = \begin{cases} 5, & \text{if } m \geq 4 \\ 4, & \text{if } m = 3, \text{ or } m = 2 \text{ and } n \geq 4 \\ 3, & \text{if } m = 2 \text{ and } n \leq 3. \end{cases}$$

Strong Products

Theorem Let $2 \leq m \leq n$ be integers. Then

$$\sigma(P_m \boxtimes P_n) = \begin{cases} 5, & \text{if } m \geq 4 \\ 4, & \text{if } m = 3, \text{ or } m = 2 \text{ and } n \geq 4 \\ 3, & \text{if } m = 2 \text{ and } n \leq 3. \end{cases}$$

Theorem (Neufeld and Nowakowski) If G and H are each connected, then $c(G \boxtimes H) \leq c(G) + c(H) - 1$.

Theorem For $n \geq 1$, $\sigma(K_{1,n} \boxtimes K_{1,n}) \geq n + 1$.

Theorem If $a, b \geq 1$ then $\sigma(K_{a,b} \boxtimes K_{a,b}) \geq ab$.

Lexicographic Products

Theorem If G and H are connected then $\sigma(G \circ H) \leq \sigma(G)|V(H)| + \sigma(H)$.

We note that $\delta(G \circ H) = \delta(G)|V(H)| + \delta(H)$, and so the bound given above is tight whenever $\sigma(G) = \delta(G)$ and $\sigma(H) = \delta(H)$.

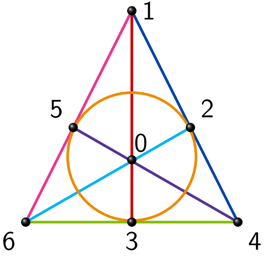
Graphs Arising from Designs

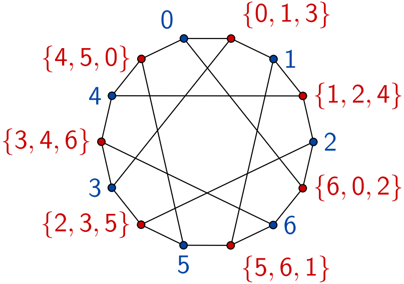
A BIBD(v, k, λ) is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of k-subsets of X called blocks such that each 2-subset of X is contained within exactly λ of the elements of $\mathcal{B}$.

A BIBD$(n^2 + n + 1, n + 1, 1)$ is known as a projective plane of order n.

The incidence graph of a BIBD(v, k, λ) $(X, \mathcal{B})$ has vertex set $X \cup \mathcal{B}$ with $x \in X$ is adjacent to $B \in \mathcal{B}$ if and only if $x \in B$.

Example





Theorem If G is the incidence graph of a projective plane with block size k, then $\sigma(G) = k + 1$.

Theorem If G is the incidence graph of a projective plane with block size k, then $\sigma(G) = k + 1$.

Theorem Let $v > k \geq 2$ be integers such that $v \equiv 0 \pmod{k}$. If G is the incidence graph of a resolvable BIBD$(v, k, 1)$, then $\sigma(G) = \frac{v}{k} + 1$.

Given a BIBD(v, k, λ), say $(X, \mathcal{B})$, its block-intersection graph is the graph having $\mathcal{B}$ as its vertex set, and two vertices are adjacent if (as blocks) they have non-empty intersection.

Given a BIBD(v, k, λ), say $(X, \mathcal{B})$, its block-intersection graph is the graph having $\mathcal{B}$ as its vertex set, and two vertices are adjacent if (as blocks) they have non-empty intersection.

Theorem If G is the block-intersection graph of a BIBD$(v, k, 1)$, then $\sigma(G) = k(r-1)$.

Proof idea:

Relabel the BIBD so that it has blocks $\{1, 2, \ldots, k\}$, $\{k+1, k+2, \ldots, 2k\}$.

Initially place the $k(r-1)$ cops on all blocks adjacent to $\{1, 2, \ldots, k\}$.

Let A be the set of blocks adjacent to $\{1, 2, \ldots, k\}$ in G, and similarly for B. Each $x \in A$ contains exactly one element in $\{1, 2, \ldots, k\}$ and each $y \in B$ contains exactly one element in $\{k+1, k+2, \ldots, 2k\}$; thus $|A \cap B| = k^2$.

Construct bipartite graph G': $\mathrm{A}' = \{x_A : x \in A\}$ and $B' = \{y_B : y \in B\}$. Let $x_A y_B$ be an edge in G' if $x_A \in A'$, $y_B \in B'$ and $x \simeq y$ in G.

Define $H \subseteq G'$: for each $i \in \{1, 2, \ldots, k\}$ include $x_A y_B \in E(G')$ such that $i \in x$, $i + k \notin x$, and $i + k \in y$. Also include $x_A y_B$ where $\{i, i+k\} \subseteq x$ and $\{i+k, j\} \subseteq y$ for $j \in \{1, 2, \ldots, k\} \setminus \{i\}$.

Clearly, G' is a $(k-1)$-regular bipartite graph so $\exists$ a 1-factor $\mathcal{F}$ of G'.

For each edge $x_A y_B$ in $\mathcal{F}$, the cops move from x to y in G thereby surrounding the robber.

Generalized Petersen Graphs

Let $A = \{a_0, \ldots, a_{n-1}\}$ and $B = \{b_0, \ldots, b_{n-1}\}$. For positive integers n and k such that $n > 2k$, the generalised Petersen graph $GP(n,k)$ has vertex set $A \cup B$ and three types of edges: for each $i \in \{0, \ldots, n-1\}$,

- a_i, a_{i+1},
- a_i, b_i, and
- b_i, b_{i+k},

with subscripts computed modulo n.

Observe that the well known Petersen graph is $GP(5,2)$.

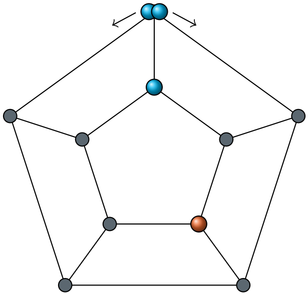
$\sigma(GP(n,k))$		k								
		1	2	3	4	5	6	7	8	9
n	3	3								
	4	3								
	5	3	3							
	6	3	3							
	7	3	4	4						
	8	3	3	3						
	9	3	4	3	4					
	10	3	4	4	4					
	11	3	4	4	4	4				
	12	3	4	3	4	4				
	13	3	4	4	4	4	4			
	14	3	4	4	4	4	4			
	15	3	4	4	4	4	4	4		
	16	3	4	4	4	4	4	4		
	17	3	4	4	4	4	4	4	4	
	18	3	4	4	4	4	4	4	4	
	19	3	4	4	4	4	4	4	4	4
	20	3	4	4	4	4	4	4	4	4

Table: Surrounding Copnumbers of Generalised Petersen Graphs, $\sigma(GP(n,k))$

Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

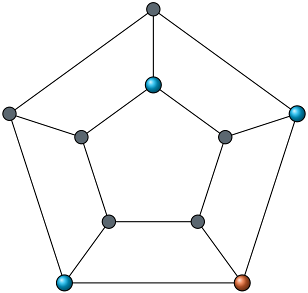
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



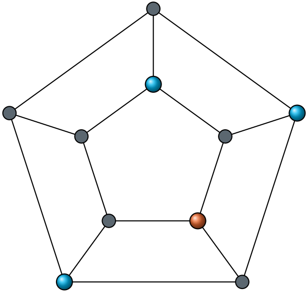
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



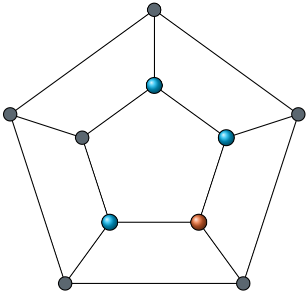
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



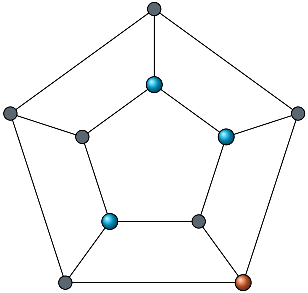
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



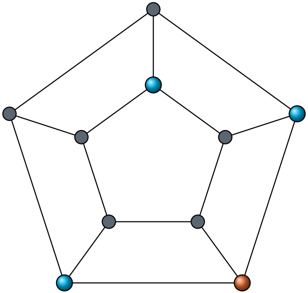
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



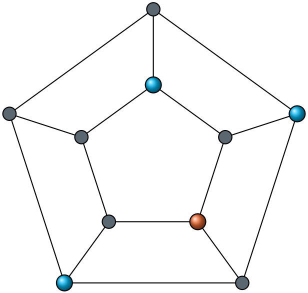
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



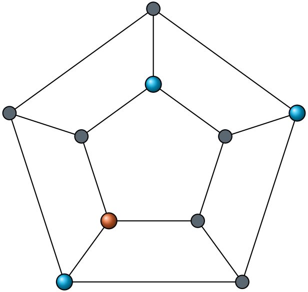
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



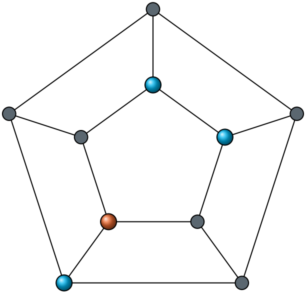
Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



Theorem For all integers $n \geq 5$ and $k \geq 2$, $\sigma(GP(n,k)) \leq 4$.

Theorem For all $n \geq 3$, $\sigma(GP(n,1)) = 3$.



Meyniel's Conjecture

Meyniel's Conjecture Let G be a graph. Then $c(G) \in O(\sqrt{n})$, where n denotes the order of G.

It is clear that there is no analogy of this conjecture that would apply to the surrounding copnumber: $\sigma(K_n) = n - 1$.

Do graphs with high surrounding copnumber inherently possess some property which in turn implies that the copnumber is low?

Open Problems

1. Our empirical results suggest that $\sigma(GP(n,k)) = 4$ whenever $n > 12$ and $k > 1$. True? Proof?

Open Problems

1. Our empirical results suggest that $\sigma(GP(n,k)) = 4$ whenever $n > 12$ and $k > 1$. True? Proof?
2. We know that $\sigma(G) > \delta(G)$ whenever $\delta(G) \geq 3$ and $g \geq 7$. Does a generalisation of this theorem, similar to the result of Frankl, hold?

Open Problems

1. Our empirical results suggest that $\sigma(GP(n,k)) = 4$ whenever $n > 12$ and $k > 1$. True? Proof?
2. We know that $\sigma(G) > \delta(G)$ whenever $\delta(G) \geq 3$ and $g \geq 7$. Does a generalisation of this theorem, similar to the result of Frankl, hold?
3. Characterise graphs G such that $\sigma(G) = k$.

Open Problems

1. Our empirical results suggest that $\sigma(GP(n,k)) = 4$ whenever $n > 12$ and $k > 1$. True? Proof?
2. We know that $\sigma(G) > \delta(G)$ whenever $\delta(G) \geq 3$ and $g \geq 7$. Does a generalisation of this theorem, similar to the result of Frankl, hold?
3. Characterise graphs G such that $\sigma(G) = k$.

It is not true that G being a connected subgraph of a connected graph H implies that $\sigma(G) \leq \sigma(H)$.

5. Under what conditions on G and e is $\sigma(G - e) \leq \sigma(G)$?

Open Problems

1. Our empirical results suggest that $\sigma(GP(n,k)) = 4$ whenever $n > 12$ and $k > 1$. True? Proof?
2. We know that $\sigma(G) > \delta(G)$ whenever $\delta(G) \geq 3$ and $g \geq 7$. Does a generalisation of this theorem, similar to the result of Frankl, hold?
3. Characterise graphs G such that $\sigma(G) = k$.

It is not true that G being a connected subgraph of a connected graph H implies that $\sigma(G) \leq \sigma(H)$.

5. Under what conditions on G and e is $\sigma(G - e) \leq \sigma(G)$?
6. Let G/e be the graph obtained by contracting the edge e. Under what conditions on G and e is $\sigma(G/e) \leq \sigma(G)$?

Thanks

Natural Sciences and Engineering Research Council of Canada
Conseil de recherches en sciences naturelles et en génie du Canada
Canada

Thanks to my co-author Andrea Burgess, who drew many of the graphs!