On the Buratti-Horak-Rosa (BHR) Conjecture for Small Supports

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Outline

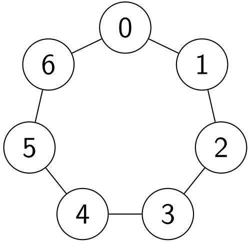
Outline

1b. Introduction to the Buratti-Horak-Rosa Conjecture

Initial Problem (by Marco Buratti)

Listing all the elements in $\mathbb{Z}_v$ with given (multiset of) $v-1$ minimal cyclic step-lengths.

Example: Two step-lengths of $\bar{1}$, three step-lengths of $\bar{2}$, one step-length of $\bar{3}$ in $\mathbb{Z}_7$.



Given multiset: $L = \{\bar{1}, \bar{1}, \bar{2}, \bar{2}, \bar{2}, \bar{3}\} = \{1, 1, 2, 2, 2, 3\} = \{1^2, 2^3, 3^1\}$

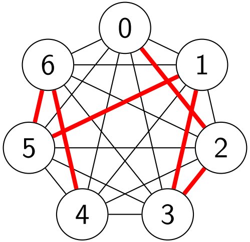
A list with such step-lengths: $[0, 2, 3, 1, 5, 6, 4]$

1b. Introduction to the Buratti-Horak-Rosa Conjecture

Equivalent Problem (in terms of Hamiltonian paths in complete graphs)

Finding Hamiltonian paths in K_v with given (multiset of) $v-1$ cyclic edge-lengths.

Corresponding Example: Multiset $L = \{1^2, 2^3, 3^1\}$ in K_7.



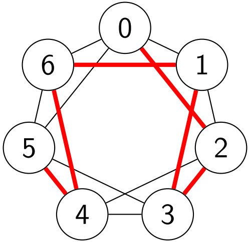
Corresponding path with such edge-lengths: $\mathbf{g} = [0, 2, 3, 1, 5, 6, 4]$.

"Definition": Underlying set / support of L: $U = \mathrm{supp}(L) = \{1, 2, 3\}$.

Equivalent Problem (in terms of Hamiltonian paths in circulant graphs)

For L with $\mathrm{supp}(L) = U$, finding Hamiltonian paths in $C_{|L|+1}(U)$ with cyclic edge-lengths as L.

A Different Example: Multiset $L = \{1^2, 2^4\}$ in $C_7(\{1,2\})$.



Corresponding path with such edge-lengths: $\mathbf{h} = [0, 2, 3, 1, 6, 4, 5]$.

Ordered edge-lengths: $(2, 1, 2, 2, 2, 1)$, i.e. a 2-edge, a 1-edge, three 2-edges, and a 1-edge.

Possible interdisciplinary applications and/or connections (of various strength):

- Graph Theory:
  Cyclic decomposition (into Hamiltonian paths) of Cayley (multi)graphs
  (Formulated by Anita Pasotti and Marco Pellegrini in 2014)
- Operations Research:
  Routing problems
- Music Theory:
  Constructing melodies and rhythms
- Electrical and Computer Engineering:
  Grid graphs (see "Construction Methods")

1c. The Buratti-Horak-Rosa Conjecture

For L multiset of $v - 1$ positive integers not exceeding $\lfloor v/2 \rfloor$

a. If v is prime, then K_v has a Hamiltonian path with cyclic edge-lengths as L (i.e. L has a **realization**).

   (Buratti's Conjecture – by Marco Buratti in 2007)

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c. There is a realization for L if and only if for any divisor d of v, the number of multiples of d in L is at most $v-d$.

   (BHR Conjecture – by Peter Horak and Alexander Rosa in 2009, reformulated by Anita Pasotti and Marco Pellegrini in 2014)

1c. The Buratti-Horak-Rosa Conjecture

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b. If $\gcd(v, x) = 1 \; \forall x \in L$, then K_v has a Hamiltonian path with cyclic edge-lengths as L.

   (Coprime BHR Conjecture – by A., Ollis in 2024)

c. There is a realization for L if and only if for any divisor d of v, the number of multiples of d in L is at most $v - d$.

   (BHR Conjecture – by Peter Horak and Alexander Rosa in 2009, reformulated by Anita Pasotti and Marco Pellegrini in 2014)

Related Questions

2007: Replace Hamiltonian paths with spanning trees.
(via Douglas West, likely asked by Marco Buratti or Jeff Dinitz)

See Ben Seamone and Brett Stevens (2012).

2012: Replace $\mathbb{Z}_v$ with arbitrary groups.
(Asked by Ben Seamone and Brett Stevens)

A., M. A. Ollis: Partial results for dihedral groups and certain Abelian groups (unsubmitted).

2014: Replace cyclic edge-lengths with other labellings.
(Asked by Marco Buratti and Anita Pasotti)

A., M. A. Ollis, Stefan Trandafir: Partial results for edge labelings from Walecki decomposition (2025+).

1d. Known results on the Buratti-Horak-Rosa Conjecture (before 2023)

$|U| = 1$: Trivial.

$|U| = 2$: Solved!

For prime v: Jeff Dinitz and Susan Janiszewski (2009)

For prime v AND also all v: Peter Horak and Alexander Rosa (HR) (2009)

$|U| = 3$: Complete solutions for specific choices of U:

Stefano Capparelli and Alberto Del Fra (2010):
$\{1, 2, 3\}$

Anita Pasotti and Marco Pellegrini (PP) (2014):
$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 8\}, \{1, 3, 5\}$

M. A. Ollis, A. Pasotti, M. Pellegrini, John Schmitt (OPPS) (2022):
$\{1, 4, 5\}$

Pranit Chand and M. A. Ollis (2022):
$\max(U) \leq 7$

1d. Known Results on the Buratti-Horak-Rosa Conjecture (before 2023)

$|U| = 3$: [Listed are some of the significant results] [Note the prevalence of 1-edges]
Partial families for many U:

PP (2014):

$L = \{1^a, 2^b, x^c\}$ when x is even and $a + b \geq x - 1$

Adrian Vazquez Avila (2022):

$L = \{1^a, 2^{x-1}, x\}$

OPPS (2021):

$L = \{1^a, x^b, (x+1)^c\}$ when x is odd and
either $a \geq \min(3x - 3, b + 2x - 3)$ or $a \geq 2x - 2$ and $c \geq 4b/3$

$L = \{1^a, x^b, (x+1)^c\}$ when x is even and
either $a \geq \min(3x - 1, c + 2x - 1)$ or $a \geq 2x - 1$ and $b \geq c$

$L = \{1^a, x^b, y^c\}$ when x is even, $x < y$ and
either y is even and $a \geq y - 1$ or y is odd and $a \geq 3y - 4$

HR (2009), followed by OPPS (2022):

$L = \{1^a, x^b, y^c\}$ where $x < y$ and $a \geq x + 4y - 5$

1d. Known Results on the Buratti-Horak-Rosa Conjecture

$|U| > 3$: [Listed are some of the significant results] [Note the prevalence of 1-edges]

HR (2009):

$v = 2m + 1$ is prime and $L = \left(\{1^2, 2^2, \ldots, m^2\} \cup \{x\}\right) \setminus \{y\}$

HR (2009), followed by OPPS (2021):

$L \cup \{1^s\}$ for some constant s depending only on U

PP (2014), followed by OPPS (2021):

$U = \{1, 2, 3, 4\}, \{1, 2, 3, 5\}$

OPPS (2021):

Partial results for $U \subseteq \{1, 2, 4, \ldots, 2x, 2x + 1\}$;

$\{1^{a_1}, 2^{a_2}, \ldots, x^{a_x}\}$ when $a_1 \geq a_2 \geq \cdots \geq a_x$

Brendan McKay and Tim Peters (2022 via Computer Search):

$|L| < 37$

1e. Results from 2023 on the Buratti-Horak-Rosa Conjecture

$|U| = 3$: A., Ollis (Announced: 2023; Submitted: 2024):

Coprime BHR Conjecture holds for the following

$U = \{1, x, x+1\}$ when $v \geq 2x^2 + 13x + 11$

$U = \{1, 2, x\}$ when $v > 4x$
(except possibly when x is odd and $a \in \{1, 2\}$ for $L = \{1^a, 2^b, x^c\}$)

Significance

Infinitely many U for which there are infinitely many values of v where the conjecture holds.
Likely a first step towards solutions for infinitely many U.
Similar techniques also provide partial families for more U.

$|U| > 3$: A., Ollis (Announced: 2023; Submitted: 2024):

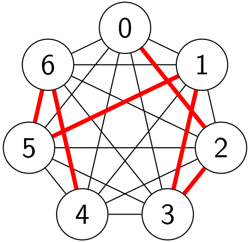
$L = \{1^a, x^b, (x+1)^c, y^d\}$ when $a \geq x + y + 1$

$L = \{1^a, x^b, (x+1)^c, y^d, (y+1)^e\}$ when $a \geq x + y + 2$

Outline

2a. Different Types of Realizations

Back to Our Initial Example: Multiset $L = \{1^2, 2^3, 3^1\}$ in K_7.



Once a realization is found, it gives rise to many others:

The realization $\mathbf{g}$:	$\mathbf{g} = [0, 2, 3, 1, 5, 6, 4]$
Reverse of $\mathbf{g}$:	$\mathbf{g}^r = [4, 6, 5, 1, 3, 2, 0]$
Complement of $\mathbf{g}$:	$\mathbf{g}^c = [6, 4, 3, 5, 1, 0, 2]$
Complement of the reverse of $\mathbf{g}$:	$(\mathbf{g}^r)^c = (\mathbf{g}^c)^r = [2, 0, 1, 5, 3, 4, 6]$

Plus, all translations of all four!

Induced Realizations from the Given Realization for $L = \{1^2, 2^3, 3^1\} = \{1, 1, 2, 2, 2, 3\}$

Directed Edge Labels	+2	+1	-2	-3	+1	-2		
Our realization, **g**	0	2	3	1	5	6	4	
Translations	1	3	4	2	6	0	5	
	2	4	5	3	0	1	6	
	3	5	6	4	1	2	0	*"Linear"*
	4	6	0	5	2	3	1	
	5	0	1	6	3	4	2	
	6	1	2	0	4	5	3	

Directed Edge Labels	+2	-1	+3	+2	-1	-2		
Reverse of **g**	4	6	5	1	3	2	0	
Translations	5	0	6	2	4	3	1	
	6	1	0	3	5	4	2	
	0	2	1	4	6	5	3	*"Standard"*
	1	3	2	5	0	6	4	
	2	4	3	6	1	0	5	
	3	5	4	0	2	1	6	

Directed Edge Labels	-2	-1	+2	+3	-1	+2		
Complement of **g**	6	4	3	5	1	0	2	
Translations	5	3	2	4	0	6	1	
	4	2	1	3	6	5	0	
	3	1	0	2	5	4	6	*"Linear"*
	2	0	6	1	4	3	5	
	1	6	5	0	3	2	4	
	0	5	4	6	2	1	3	

Directed Edge Labels	-2	+1	-3	-2	+1	+2		
Complement & Reverse	2	0	1	5	3	4	6	
Translations	1	6	0	4	2	3	5	
	0	5	6	3	1	2	4	
	6	4	5	2	0	1	3	*"Linear"*
	5	3	4	1	6	0	2	
	4	2	3	0	5	6	1	
	3	1	2	6	4	5	0	

Other Realizations for the Same *L*

Different Directed Edge Labels	+2	+1	-2	-3	-1	+2		
Another Realization	0	2	3	1	5	4	6	***etc.***

A Special Type of Realization (for a Different *L*)

Different Directed Edge Labels	+2	-1	+3	+1	-2	**+3**		
A Special Realization	0	2	1	4	5	3	6	*"Perfect"*

2b. Main Result for $U = \{1, x, x+1\}$

Let $x > 1$.

Construction

A standard linear realization is constructed for $L = \{1^a, x^b, (x+1)^c\}$ whenever $a \geq x + 1$.

Theorem

For all $v \geq 2x^2 + 13x + 11$*, the Coprime BHR Conjecture holds for* $U = \{1, x, x+1\}$*.*

The theorem follows from the construction via straightforward modular arithmetic, as demonstrated by the following example.

2b. Main Result for $U = \{1, x, x+1\}$

Example

Let $v = 101$ and let $U = \{1, 7, 8\}$.

In mod 101:

$7^{-1} = 29$ and $\{1, 7, 8\} \xmapsto{\times 29} \{29, 1, 30\}$

$8^{-1} = 38$ and $\{1, 7, 8\} \xmapsto{\times 38} \{38, 64, 1\} \equiv \{38, 37, 1\}$

Thus, $\{1^a, 7^b, 8^c\}$, $\{29^a, 1^b, 30^c\}$, and $\{38^a, 37^b, 1^c\}$ are equivalent.

Hence, we can construct a realization whenever $a \geq 8$, $b \geq 30$, or $c \geq 38$.

Therefore, a counterexample would need $a + b + c < 8 + 30 + 38 = 76$.

This contradicts with $a + b + c = v - 1 = 100$.

2c. Expanded Results since 2024

Let $1 < x < y$ with $k = y - x$.

Construction

$k \leq 2$: A linear realization is constructed for $L = \{1^a, x^b, y^c\}$ whenever $a \geq y$.

$k = 3$: A linear realization is constructed for $L = \{1^a, x^b, y^c\}$ whenever $a \geq y + 2$.

$k = 1$: Result from 2023.

$k = 2$: Announced mid-2024, submitted early 2025 (to CODESCO'24 proceedings). When combined with OPPS (2021), gives a standard linear realization.

$k = 3$: Announced March 2025, to-be-submitted in 2025 (to PYTH5 proceedings?). The construction for $k = 3$ yields a standard linear realization.

2d. Other Constructions since 2024

Let $L = \{1^a, x^b, y^c\}$ with $1 < x < y$.
A linear realization is constructed for L in the following cases:

- $a \geq x + y - 1$,
- $a \geq x + y - 2$ when $x = 3$ or x is even,
- $a \geq 4x - 3$ for x odd, $y > 2x - 2$, and $b \geq y - 2x + 2$,
- $a \geq x$ for $y = tx$, with x and t odd, and $b \geq tx + 2t - 3$,
- $a \geq 7$ for $x = 3$ and $b \geq y - 4$.
- x even, $y \geq 2x + 1$, $a \geq 3x - 2$ and $a + b \geq x + y - 1$,
- $x = 2$, $y \geq 5$ and $a \geq 3$.

2e. Select BHR Implications since 2024

For a multiset L of size $v-1$ with $\mathrm{supp}(L)=\{1,x,y\}$, the Coprime BHR Conjecture holds in the following cases:

- $y=x+1$, $x\leq 15$
- $x>2$, $y>2x+19$, and v is sufficiently large,
- $x=3$, $y=3t$, and $v>6t+53$,
- $x,t\geq 7$, $y=tx$.

(Main tool enabling all these results: **GRID GRAPHS**)

Outline

3a. Tool 1: Circulant Graphs have associated Toroidal Lattices

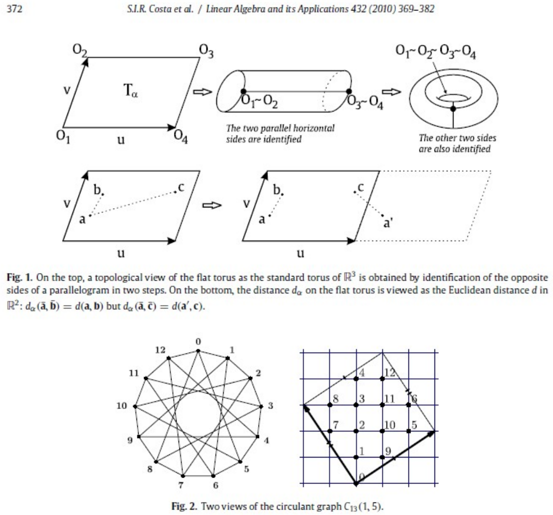

372 S.I.R. Costa et al. / Linear Algebra and its Applications 432 (2010) 369–382

Fig. 1. On the top, a topological view of the flat torus as the standard torus of $\mathbb{R}^3$ is obtained by identification of the opposite sides of a parallelogram in two steps. On the bottom, the distance d_α on the flat torus is viewed as the Euclidean distance d in $\mathbb{R}^2$: $d_\alpha(\bar{\mathbf{a}}, \bar{\mathbf{b}}) = d(\mathbf{a}, \mathbf{b})$ but $d_\alpha(\bar{\mathbf{a}}, \bar{\mathbf{c}}) = d(\mathbf{a}', \mathbf{c})$.

Fig. 2. Two views of the circulant graph $C_{13}(1, 5)$.

Source for the figures:

S.I.R. Costa, J.E. Strapasson,
M.M.S. Alves, T.B. Carlos,

Circulant graphs and tessellations on flat tori,

Linear Algebra and its Applications
Volume 432, Issue 1,
1 January 2010,
Pages 369-382

doi:10.1016/j.laa.2009.08.040

3b. Technique 1: Standard Linear Realizations for $U = \{1, x\}$

$\omega(x, b)$: the smallest possible number of 1-edges used in a standard linear realization with b x-edges and support $U = \{1, x\}$

Note 1: Inspired by some of the constructions by Jeff Dinitz and Susan Janiszewski (2009), as well as Peter Horak and Alexander Rosa (2009). However those papers were not concerned with minimizing the number of 1-edges.

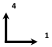
Note 1: For given $b \geq 0$ and $x > 1$,
$\omega(x, b)$ depends on the Euclidean division of b by x
(i.e. the unique integers q and r such that $b = qx + r$ and $0 \leq r < x$).

Method: To find $\omega(x, b)$, we work on the tiling of the plane by the associated toroidal lattice of $C_v(U)$, where vertex labels differ horizontally by 1 and vertically by x.

Example: Let's determine $\omega(4, 14)$.

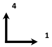
Start with an integer lattice with horizontal steps of size 1 and vertical steps of size x = 4

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	**0**	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
-26	-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1
-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5



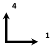
Re-label the lattice modulo v = 14 *(Redundant in case of linear realizations)*

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



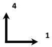
Linear realizations are restricted to a slanted lattice

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



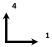
Standard linear realizations are further restricted from the left by a zig-zag path…

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



Standard linear realizations are further restricted from the left by a zig-zag path...

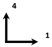
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



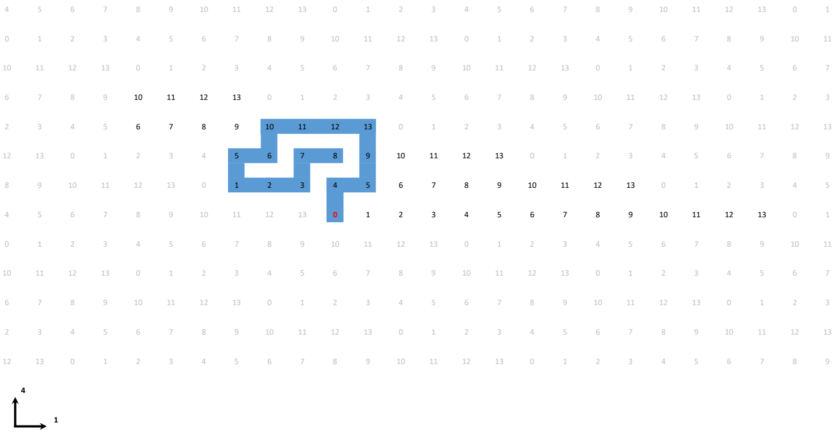
...And from below by the path of 1s

Building an example of a standard linear realization

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9

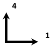


Building an example of a standard linear realization



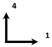
On the other hand, minizing *a* (the number of 1-edges) limits a linear realization to a bridging of *"fauxsets"*

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	**13**	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	**9**	**10**	**11**	**12**	**13**	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	**5**	**6**	**7**	**8**	**9**	**10**	**11**	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	**0**	**1**	**2**	**3**	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



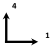
An example of a (non-standard) linear realization with minimal *a*

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



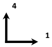
The 2 conditions combined restricts the *"fauxsets" to-be-bridged to a stack either to the right of 0…*

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



The 2 conditions combined restricts the *"fauxsets" to-be-bridged to a stack either to the right of 0...*

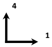
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



...Or to the left of 0, with 4 fauxsets to be bridged, implying that ω (4, 14) is at least 3.

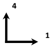
In our example, if we pick left, we fail!

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



If we pick right, we succeed!

4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5
4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11
10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7
6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3
2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9



Thus, $\omega(4, 14) = 3$

3b. Technique 1: Standard Linear Realizations for $U = \{1, x\}$

Theorem

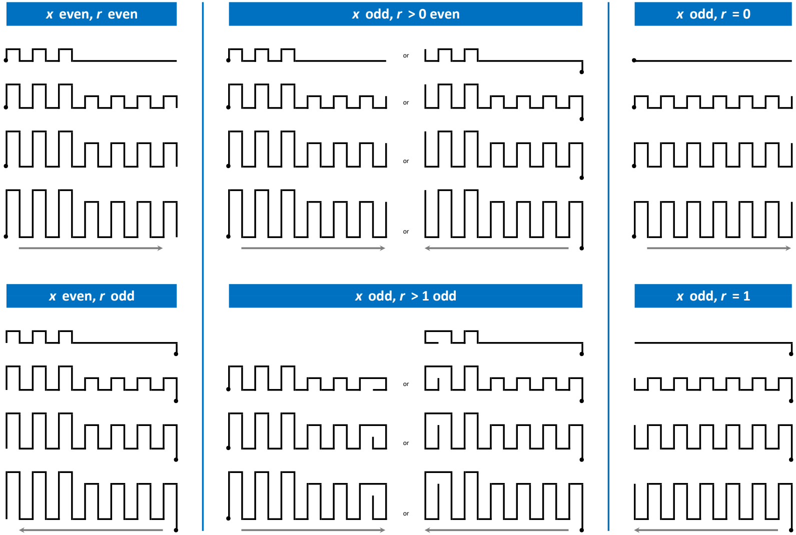
For given $b \geq 0$ and $x > 1$ with Euclidean division of $b = qx + r$,

$$\omega(x, b) = \begin{cases} x, & \text{if } x \text{ and } r \text{ are both odd and } r > 1 \text{ with } q > 0 \\ x - 1, & \text{otherwise.} \end{cases}$$

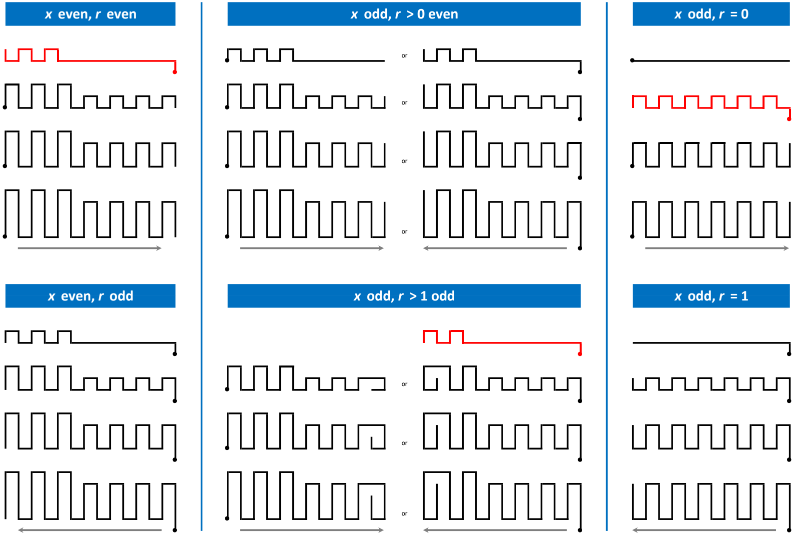
Note 1: The constructions yield **growable** realizations.

Note 2: The exceptional case has an additional 1-edge due to a "tail-curl" that is needed.

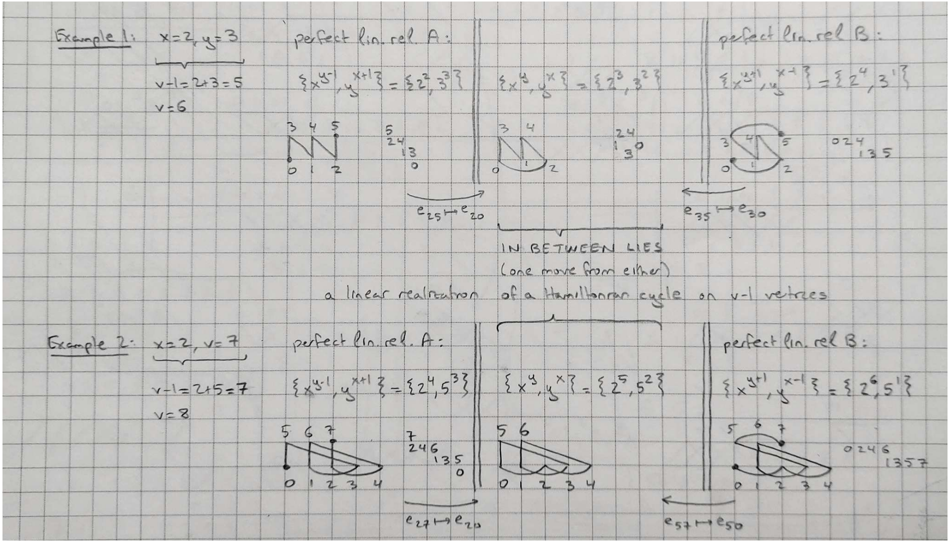
Optimal Constructions (i.e. minimal a) for Standard Linear Realizations of the Multisets $\{1^a, x^c\}$, where $c = qx + r$ with $r < x$



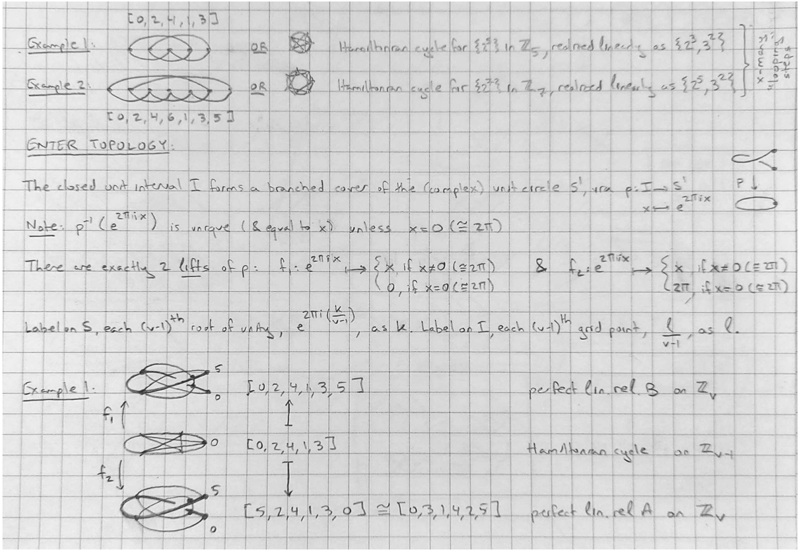
Optimal Constructions (i.e. minimal a) for Standard Linear Realizations of the Multisets $\{1^a, x^c\}$, where $c = qx + r$ with $r < x$



3c. Tool 2: Branched Covers yield Perfect Linear Realizations



3c. Tool 2: Branched Covers yield Perfect Linear Realizations



3d. Technique 2: Core Perfect Realizations for $|U| = 2$

Construction

A perfect linear realization is constructed for $L = \{x^{y-1}, y^{x+1}\}$ for coprime x and y.

Immediate Corollary

A perfect linear realization is constructed for $L = \{x^x, (x+1)^{x+1}\}$.

Other Corollaries

A standard linear realization is constructed for $L = \{1^{k-1}, x^{jx}, (x+k)^{j(x+k)}\}$ for all $j \geq 1$ when $k < 4$ or $k|y$. In the latter case, if k is odd, then the realization is perfect.

A standard linear realization is constructed for $L = \{1^{\gcd(x,y)}, x^{2yj}, y^{2xj}\}$ for all $j \geq 1$. If $\gcd(x, y)$ is odd, then the realization is perfect.

3e. Tool 3: Path Concatenations

$\mathbf{g} \oplus \mathbf{h}$: The complement of $\mathbf{g}$ (with v vertices) and the translation of $\mathbf{h}$ by $v-1$, identified at the end-vertices labeled with $v-1$.

Use: Realizations on K_v and K_w to a realization on K_{v+w-1}:

1. standard $\oplus$ standard $\implies$ linear
2. standard $\oplus$ perfect $\implies$ standard
3. perfect $\oplus$ perfect $\implies$ perfect

Note: For the multiset $\{1^s\}$, $[0, 1, \ldots, s-1]$ is a perfect realization.

Lemma: If L has a standard (respectively perfect) realization then $L \cup \{1^s\}$ has a standard (respectively perfect) realization $\forall s \geq 0$.

Technique

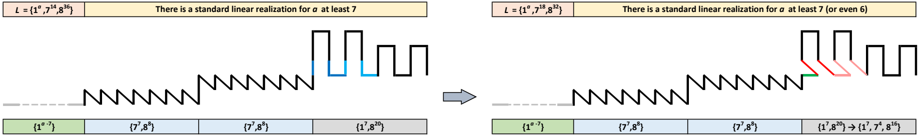
To construct realizations with $a \geq x + 1$ for $L = \{1^a, x^b, (x+1)^c\}$ with given b and c, use concatenated perfect realizations to reduce to b' and c', followed by modifications on the optimal constructions for either $U = \{1, x\}$ or $U = \{1, x+1\}$, as needed.

Constructing realizations with $a \geq 7$ for $L = \{1^a, 7^{18}, 8^{32}\}$

$$\{1^a, 7^{18}, 8^{32}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^4, 8^{16}\}$$

Since 4 < 16, we use modifications on the optimal construction for $\{1^a, 8^{4+16}\}$:

$$\{1^7, 8^{4+16}\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} = \{1^7, 7^4, 8^{16}\}$$

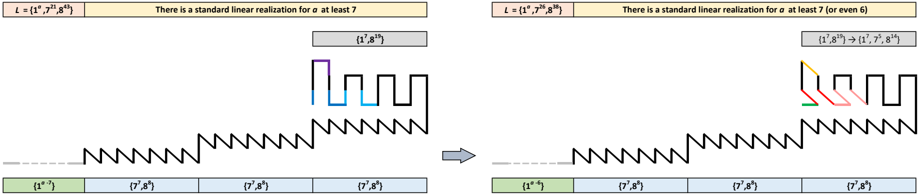


Constructing realizations with $a \geq 6$ for $L = \{1^a, 7^{26}, 8^{38}\}$

$$\{1^a, 7^{26}, 8^{38}\} = \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{7^7, 8^8\} \cup \{1^a, 7^5, 8^{14}\}$$

Since 5 < 14, we use modifications on the optimal construction for $\{1^a, 8^{5+14}\}$:

$$\{1^7, 8^{5+14}\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^2\} \cup \{1^1, 7^2\} \setminus \{1^1, 8^1\} \cup \{7^1\} = \{1^6, 7^5, 8^{14}\}$$

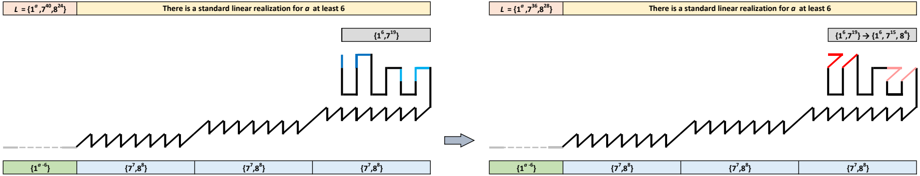


Constructing realizations with $a \geq 6$ for $L = \{1^{a}, 7^{36}, 8^{28}\}$

$$\{1^{a}, 7^{36}, 8^{28}\} = \{7^{7}, 8^{8}\} \cup \{7^{7}, 8^{8}\} \cup \{7^{7}, 8^{8}\} \cup \{1^{a}, 7^{15}, 8^{4}\}$$

Since 15 > 4, we use modifications on the optimal construction for $\{1^{a}, 7^{15+4}\}$

$$\{1^{6}, 7^{15+4}\} \setminus \{1^{1}, 7^{2}\} \cup \{1^{1}, 8^{2}\} \setminus \{1^{1}, 7^{2}\} \cup \{1^{1}, 8^{2}\} = \{1^{6}, 7^{15}, 8^{4}\}$$

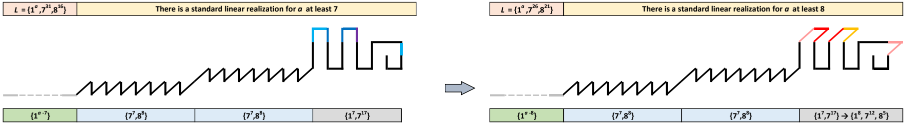


Constructing realizations with $a \geq 8$ for $L = \{1^{a}, 7^{26}, 8^{21}\}$

$$\{1^{a}, 7^{26}, 8^{21}\} = \{7^{7}, 8^{8}\} \cup \{7^{7}, 8^{8}\} \cup \{1^{a}, 7^{12}, 8^{5}\}$$

Since 12 > 5, we use modifications on the optimal construction for $\{1^{a}, 7^{12+5}\}$

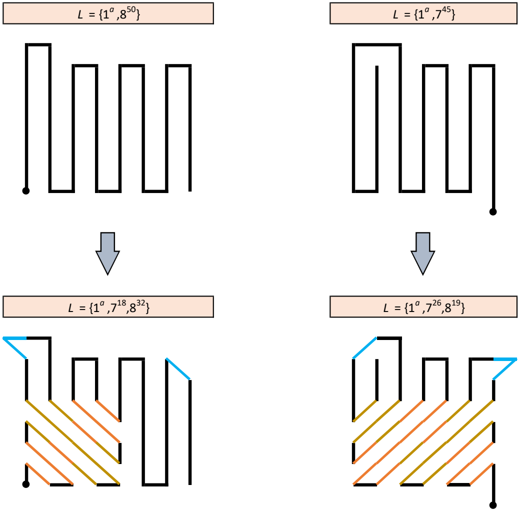
$$\{1^{7}, 7^{12+5}\} \setminus \{1^{1}, 7^{2}\} \cup \{1^{1}, 8^{2}\} \setminus \{1^{1}, 7^{2}\} \cup \{1^{1}, 8^{2}\} \setminus \{7^{1}\} \cup \{1^{1}, 8^{1}\} = \{1^{8}, 7^{15}, 8^{4}\}$$



Outline

Strategy

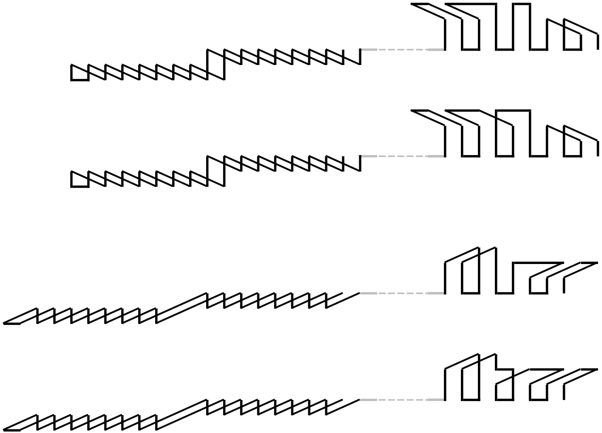
Use block modifications on the optimal constructions for either $U = \{1, x\}$ (if $a \geq b$) or $U = \{1, x+1\}$ (if $b \geq$ a).


$L = \{1^a, 8^{50}\}$
$L = \{1^a, 7^{45}\}$
$L = \{1^a, 7^{18}, 8^{32}\}$
$L = \{1^a, 7^{26}, 8^{19}\}$

Technique for $k \leq 3$

To construct realizations with $a \geq x + k$ for $L = \{1^a, x^b, (x+k)^c\}$ with given b and c, use "concatenated" spanning linear forests to reduce to b' and c', followed by modifications on the optimal constructions for either $U = \{1, x\}$ or $U = \{1, x+k\}$, as needed.

Examples of Concatenated Constructions for U = {1, *y* - 2, *y* }



4b. Extensions to Other Cases of $|U| = 3$

Conjectures

1. A concatenated linear realization can be constructed for $L = \{1^a, x^b, y^c\}$ whenever $a \geq y + \gcd(x, y) - 1$.
2. A non-concatenated standard linear realization can be constructed for $L = \{1^a, (y-k)^b, y^c\}$ and $L = \{1^a, k^b, y^c\}$ whenever $a \geq y$, for $k \leq \lfloor v/2 \rfloor$.

4c. Concluding Remarks

THANK YOU!

QUESTIONS / COMMENTS?