Classifying Generalized Howell Designs

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Important Combinatorial Spaces

1. Hamming space
2. Johnson space
3. Grassmannian space
4. Permutation space

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Generalized Howell Designs

Definition

A t-GHD$_k(s, v; \lambda)$ generalized Howell design is an $s \times s$ array, each cell of which is either empty or contains a k-subset of elements of some set X of size v such that (i) each element of X appears exactly once in each row and in each column and (ii) no t-subset of elements from X appears in more than λ cells.

Example. A 2-GHD$_3(4, 12; 1)$. $X = \{$a,b,c,d,e,f,g,h,i,j,k,l$\}$.

aef	bgh	cij	dkl
bik	ajl	deg	cfh
cgl	dfi	ahk	bej
dhj	cek	bfl	agi

Another Example

Example. A 2-$GHD_3(8, 18; 1)$.

		agh	bkm	ejp	fir	clq	dno
		bln	aij	cho	dgq	ekr	fmp
acd	boq			gmr	hkp	fjn	eil
bpr	aef			inq	jlo	dhm	cgk
fgl	dik	emq	cnr			aop	bhj
ehn	cjm	fko	dlp			bgi	aqr
imo	gnp	djr	fhq	akl	bce		
jkq	hlr	cip	ego	bdf	amn		

The empty cells define an $(s - v/k)$-regular bipartite graph (degree $8 - 18/3 = 2$ here).

Generalized Howell Designs

Some subclasses of t-$\text{GHD}_k(s, v; \lambda)$ Howell designs:

Room square	$2\text{-GHD}_2(v-1, v; 1)$
Howell design	$2\text{-GHD}_2(s, v; 1)$
doubly resolvable BIBD	$2\text{-GHD}_k(\lambda(v-1)/(k-1), v; \lambda)$
SOMA	$2\text{-GHD}_k(s, ks; 1)$
semi-Latin square	$2\text{-GHD}_k(s, ks; \infty)$
MOLS	SOMA of type $(1, \ldots, 1)$

Problem

Is there a $\mathrm{DRKTS}(21)$? $2\text{-GHD}_3(10, 21; 1)$

Problem

Are there 3 MOLS *of order* 10? $2\text{-GHD}_3(10, 30; 1)$ *of type* $(1, \ldots, 1)$

Permutation Code

permutation code An (s, d) permutation code (or permutation array) has codewords that are permutations of the elements $\{0, 1, \ldots, s-1\}$ and has Hamming distance at least d between any pair of codewords.

$(k\text{-})$uniform code The number of codewords with a given value in a given coordinate is either 0 or k.

Example. A 3-uniform $(4, 3)$ permutation code.

0123	2013
0231	2130
0312	2301
1032	3021
1203	3102
1320	3210

Generalized Howell Designs as Codes

2-GHD$_3(4, 12; 1) \leftrightarrow$ 3-uniform $(4, 3)$ permutation code of size 12:

aef	bgh	cij	dkl
bik	ajl	deg	cfh
cgl	dfi	ahk	bej
dhj	cek	bfl	agi

$\leftrightarrow$

0123	2013
0231	2130
0312	2301
1032	3021
1203	3102
1320	3210

2-GHD$_k(s, v; \lambda) \leftrightarrow$ k-uniform $(s, s - \lambda)$ permutation code of size v

Symmetries

	Generalized Howell design	Permutation codes
1	permute columns	permute columns
2	permute rows	permute values
3	transpose rows $\leftrightarrow$ columns	replace permutations by inverses

Terminology for codes:
1–2: equivalence, $|G| = (s!)^2$
1–3: isometry, $|G| = 2(s!)^2$

Classifying k-Uniform (s, d) Permutation Codes

Outline of algorithm:

1. For each constellation of empty cells = for each $(s - v/k)$-regular bicolored graph of size $2s$, carry out Steps 2 and 3.
2. Classify a set of k codewords with the same value in one coordinate.
    - Add codewords.
    - Carry out isomorph rejection.
    - Validate with double counting.
3. Extend the solutions in Step 1 to codes of size v.
    - Extend with tailored algorithm.
    - Carry out isomorph rejection.
    - Validate with double counting.

Some Details of the Classification

00224466
11335577
23460715
47061352
35762041
43657201
73506142
65143702
75610324
27156034
57403621
32571604
52746310
62417053
36047125
46512730
56170243
24673150
64701235
74052613

Isomorph rejection: canonical augmentation (canonical construction path); codes are mapped to graphs that are processed with `nauty`.

Double counting: When classifying partial codes of size M one gets (Orbit–Stabilizer Theorem):

$$\sum_{C \in \mathcal{D}} \frac{2(s!)^2}{|\mathrm{Aut}(C,F)|}$$

and from the search data one gets

$$\frac{1}{M} \sum_{C \in \mathcal{C}} \frac{2(s!)^2 \cdot e(C)}{|\mathrm{Aut}(C,F)|}.$$

Table of # of Designs

A necessary condition for the existence of a 2-$GHD_3(s, v; 1)$ is that

$$\frac{v}{3} \leq s \leq \frac{v-1}{2}.$$

$v \backslash s$	$v/3$	$v/3+1$	$v/3+2$	$v/3+3$
3	1	-	-	-
6	0	-	-	-
9	0	0	-	-
12	1	0	-	-
15	1	0	0	-
18	4	1	5	-
21	340	?	?	?: DRKTS(21)

A New Generalized Howell Design

The unique 2-$GHD_3(7, 18; 1)$:

	pyt	hag	ore	vni	cfk	bdj
bhp		cei	aky	dfr	jno	gtv
cov	dhi		gnp	jkt	abr	efy
dny	krv	fjp		bcg	eht	aio
aft	ben	dko	hjv		giy	cpr
ijr	fgo	bvy	cdt	aep		hkn
egk	acj	nrt	bfi	hoy	dpv	

The order of the automorphism group is 6.

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