# Coadjoint Matroids and Dependencies on Hypergraphs

Ragnar Freij-Hollanti, Patricija Šapokaitė

June 6, 2025

Kalamata, Greece

Aalto-yliopisto

# Table of Contents



Aalto-yliopisto

# Prerequisites and Known Results

# Foundations and Previous Work

We continue to research dependencies on matroids using the terminology defined here:

*Freij-Hollanti, Ragnar and Šapokaitė, Patricija. Matroidal Cycles and Hypergraph Families, arXiv:2410.23932*

The preprint is based on the definition of combinatorial derived matroids that was presented here:

*Freij-Hollanti, Ragnar and Jurrius, Relinde and Kuznetsova, Olga. Combinatorial Derived Matroids, The Electronic Journal of Combinatorics, 30:P2.8, 28pp*

Aalto-yliopisto

# Hypergraphs

**Definition 1.1.** *A **hypergraph** is a pair* $(V, \mathcal{E})$*, where* $V$ *is a set of* vertices *and* $\mathcal{E} \subseteq \mathcal{P}(V)$ *is a set of* hyperedges. *A hypergraph is* simple *if there are not two edges* $e, f \in \mathcal{E}$ *with* $e \subseteq f$. *A hypergraph is* $k$***-regular*** *if* $|e| = k$ *for all* $e \in \mathcal{E}$.

# Matroids via dependent sets

**Definition 1.2.** *A **matroid** is a pair $(E, \mathcal{D})$, where $E$ denotes a finite set and $\mathcal{D}$ is a family of subsets of $E$ satisfying the following conditions:*

- $\emptyset \notin \mathcal{D}$;
- $D \in \mathcal{D}$ *and* $D \subseteq D' \Rightarrow D' \in \mathcal{D}$;
- $D_1, D_2 \in \mathcal{D}$ *and* $D_1 \cap D_2 \notin \mathcal{D} \rightarrow (D_1 \cup D_2) \setminus \{e\} \in \mathcal{D}$ *for all* $e \in D_1 \cap D_2$.

The minimal dependent sets are calles **circuits**.

# Matroids via circuits

**Theorem 1.3.** *A* ***simple hypergraph*** $M = (V, \mathcal{E})$ *is a circuit hypergraph of a matroid if for every* $C_1, C_2 \in \mathcal{E}$ *with* $C_1 \neq C_2$*, and every* $v \in C_1 \cap C_2$*, there exists* $C_3 \in \mathcal{E}$ *such that* $C_3 \subseteq C_1 \cup C_2 \setminus \{v\}$*. The set* $\mathcal{E}$ *is the* circuit set *of* $M$*.*

A 2-graph is the circuit hypergraph of a matroid if and only if it is a disjoint union of cliques.

# Matroidal Hypergraph Cycles

**Definition 1.4.** *A collection of edges* $\{e_1, \ldots, e_k\}$ *in a hypergraph* $H$ *is* ***doubly covering*** *if*

$$e_i \subseteq \bigcup_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} e_j$$

*for all* $i = 1, \ldots, k$.

**Definition 1.5.** *We will call a doubly covering set that does not have any doubly covering proper subset a* ***matroidal cycle****.*

# Natural Matroids

Here we will use the notion of *naturality* as describing dependencies that feel the most *natural* when thinking about dependent pieces of information - doubly covering the vertices.

The definitions of **natural cycle**, **natural matroid** or **natural hypergraph**, will refer to the matroidal cycles satisfying the double covering.

**Definition 1.6.** *If a hypergraph does not contain a cycle, we will call it a **natural tree**.*

# $\epsilon$ OPERATION

**Definition 1.7.** *Let $\mathcal{E}$ be the edge set of a hypergraph $H = (V, \mathcal{E})$. Denote by*

$$\epsilon(\mathcal{E}) := \mathcal{E} \cup \{(A_1 \cup A_2) \setminus \{v\} :$$

$$A_1, A_2 \in \mathcal{E}, A_1 \cap A_2 \notin \mathcal{E}, v \in A_1 \cap A_2\},$$

$$\min \mathcal{E} := \{A \in \mathcal{E} : \nexists A' \in \mathcal{E} : A' \subsetneq A\},$$

*and*

$$\uparrow \mathcal{E} := \{A \in \mathcal{E} : \exists A' \in \mathcal{E} : A' \subseteq A\}.$$

# Derived matroid

**Definition 1.8.** *Consider a matroid $M$ with a collection of circuits $\mathcal{C}$. Then the* ***combinatorial derived matroid*** *$\delta M$ is a matroid, represented by the closure of the hypergraph with $E = \mathcal{C}$.*

# Rank

**Definition 1.9.** *The* ***rank of a matroidal hypergraph*** *is the largest set of vertices such that there are no edges containing only those vertices. We will call the vertices in said set a* ***basis****.*

# Closures and Coadjoints

# Closures

**Definition 2.1.** *Let $r$ be the rank function of a matroid $M$ with ground set $V$. We call the set $\mathrm{cl}(S) = \mathrm{cl}_M(S)$ the **closure** of $S$ in $M$ if*

$$\mathrm{cl}(S) = \{x \in V : r(S \cup x) = r(S)\}.$$

# Matroidal Closures

**Definition 2.2.** *Let $H = (V, E)$ be a hypergraph. Let $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{E}_{i+1} = \min \epsilon\mathcal{E}_i$ for $i \in \mathbb{N}$ and $\mathcal{E}' = \min\left(\cup_i E_i\right)$. Moreover, let $\mathcal{F}_0 = \mathcal{E}$, $\mathcal{F}_{i+1} =\uparrow \epsilon\mathcal{F}_i$ for $i \in \mathbb{N}$ and $\mathcal{F} = \left(\cup_i F_i\right)$. Then $\min \mathcal{F} = \mathcal{E}'$, and $\overline{H} = (V, \mathcal{E}')$ is a matroid. We call this the **matroidal closure** of $H$.*

Aalto-yliopisto

## $\mathcal{U}$ AND $\mathcal{F}$

For a matroid $M$ with circuit set $\mathcal{C}(M)$ and rank function $r$, we denote its lattice of cyclic sets by $\mathcal{U}(M)$ and its lattice of flats by $\mathcal{F}(M)$. This means that

$$\mathcal{F}(M) = \{F \subseteq E(M) : \mathrm{cl}(F) = F\}$$

and

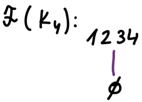
$$\mathcal{U}(M) = \{U \subseteq E(M) : r(U - x) = r(U) \text{ for all } x \in U\} =$$

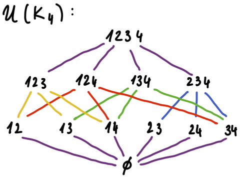
$$= \left\{ \bigcup_{C \in S} C : S \subseteq \mathcal{C}(M) \right\}$$

It is well known that $\mathcal{F}(M)$ is a geometric lattice and that $\mathcal{U}(M)$ is cogeometric, as its dual is isomorphic to $\mathcal{F}(M^*)$.

# Examples

Take a complete graph on 4 vertices, $K_4$. Then its lattice of flats and geometric lattice look as follows:



## Some Results

**Proposition 2.3.** *Let* $\mathcal{H} = (V, E)$ *and* $\delta\mathcal{H} = (E, \mathcal{C})$*. For every* $S \in E$,

$$\mathrm{cl}_{\delta M}\left(\{e \in E(\mathcal{H}) : e \in S\}\right) \supseteq \overline{S(\mathcal{H})} \cap \mathcal{H}.$$

**Proposition 2.4.** *Let* $\mathcal{H} = (V, E)$ *be a matroid. Then*

$$E(\mathrm{cl}(S)) \supseteq \overline{E(S)} \cap E(\mathcal{H})$$

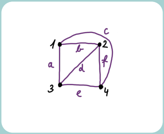
# The Goal and Main Results

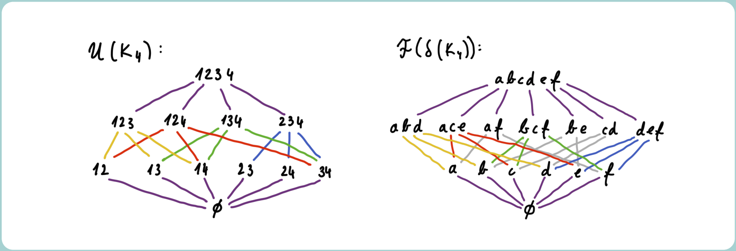
# Our Goal

**Conjecture 3.1.** *Let $M$ be a matroid. If $M$ has a coadjoint $N$, then $\delta M$ is a coadjoint of $M$.*

# Motivation

We again loot at a complete graph $K_4$. Its derived matroid $\delta(K_4)$ consists of the edges $\{abd, abef, acdf, ace, bcde, bef, def\}$ and all of the 5-vertex sets. Therefore $\mathcal{F}(\delta(K_4))$ looks like the one on the right.



## The Main Result

**Theorem 3.2.** *Let $M$ be a natural matroid. If $M$ has a coadjoint $N$, then $\delta M$ is a coadjoint of $M$.*

# Proof Steps

**Lemma 3.3.** *The map*

$$\Phi : U \mapsto \mathrm{cl}_{\delta M}\left(\{C \in \mathcal{C}(M) : C \subseteq U\}\right)$$

*is an order-preserving map* $\mathcal{U}(M) \to \mathcal{F}(\delta M)$.

**Lemma 3.4.** *If M is a natural matroid, the map*

$$\Phi : U \mapsto \mathrm{cl}_{\delta M}\left(\{C \in \mathcal{C}(M) : C \subseteq U\}\right)$$

*is injective.*

# References


- Freij-Hollanti, Ragnar and Jurrius, Relinde and Kuznetsova, Olga. Combinatorial Derived Matroids, The Electronic Journal of Combinatorics, P2.8, 28pp, 2023.
- Freij-Hollanti, Ragnar and Šapokaitė, Patricija, Matroidal Cycles and Hypergraph Families, ArXiv, https://arxiv.org/abs/2410.23932, 2024.
- Main, Roger Anthony. Hypergraphic Matroids, The Open University, 1978.