Erdős-Ko-Rado problems and Uniqueness

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joint work with
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Overview

The EKR problem

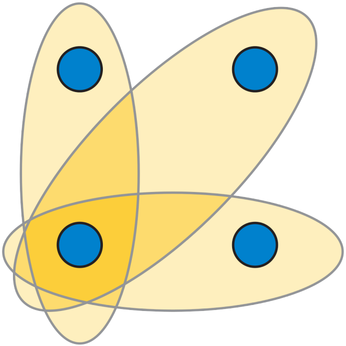

Figure: Star-shaped EKR-set [1]

[1] https://upload.wikimedia.org/wikipedia/commons/8/86/Intersecting_set_families_2-of-4.svg

Kneser graphs

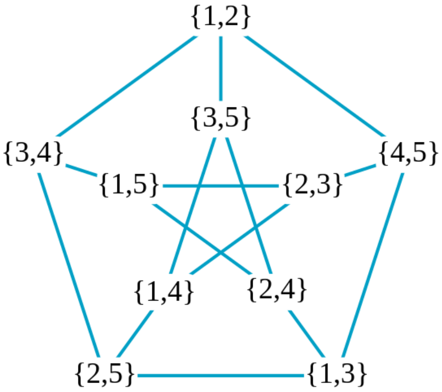

Figure: The Kneser graph $K(5,2)$ [2]

[2] https://en.wikipedia.org/wiki/File:Kneser-5-2.svg

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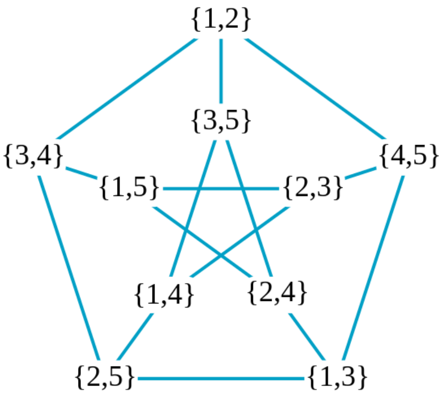

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EKR-sets are cocliques of the Kneser graph.

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The finite classical polar space $\mathrm{PS}(n, e, q)$

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For a totally isotopic subspace U, we have
$U^\perp = \{v \in \mathbb{F}_q^d \mid f(v, u) = 0 \text{ for all } u \in U\}$

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Chambers of polar spaces

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Opposition of chambers

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The EKR-problem on chambers of $\mathrm{PS}(n, e, q)$ is the following:

Let $\mathcal{F}$ be a set of pairwise non-opposite chambers.
How big can $\mathcal{F}$ be?
What is the structure of $\mathcal{F}$?

The Hoffman ratio-bound

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Theorem (Hoffman ratio-bound)

$$\alpha(\Gamma) \leq |X| \frac{-\lambda_{min}}{d - \lambda_{min}}$$

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Consider $\mathrm{PS}(n, e, q)$ with $e \geq 1$ or n even.
Let $\mathcal{F}$ be an EKR-set of chambers. Then

$$|\mathcal{F}| \leq \frac{\Phi}{q^{n+e-1} + 1}.$$

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Consider the vector space $\mathbb{R}^d$ where the entries are indexed by the chambers of $\mathrm{PS}(n, e, q)$.

Antidesigns

for $e \geq 1$ or n even

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$$\mathbb{1}_{\mathcal{F}}^\top w = \frac{\mathbb{1}^\top w}{q^{n-1+e} + 1}$$

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For a chamber $C = (C_1, \dots, C_n)$ this means

$$w_C(B) := \begin{cases} -\lambda_{min} & \text{if } C = B, \\ 1 & \text{if } C \text{ and } B \text{ are opposite,} \\ 0 & \text{otherwise.} \end{cases}$$

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Let π be a generator. We have the antidesign

$$w_\pi(B) := \sum_{C_n = \pi} w_C(B) = \begin{cases} -\lambda_{min} & \text{if } B_n = \pi, \\ q^{n(n-1)/2} & \text{if } B_n \cap \pi = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

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with $\mathbb{1}_{\mathcal{F}}^\top w_\pi = -\lambda_{min} \cdot \Phi'$, where Φ' is the number of chambers with $C_n = \pi$.

Reduction to s-spaces
via Antidesigns

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$\rightarrow$ only n possibilities left

Hoffman bound for s-spaces

for $1 < s < n$

Assume that the Hoffman ratio-bound is tight for s-spaces.

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Play them off against each other and against the geometry.

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Contradiction.

Result

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Thank you for your attention