Transitivity in weighted directed graphs

Piotr J. Wojciechowski

Department of Mathematical Sciences
The University of Texas at El Paso
USA

Joint work

- Grega Cigler
  Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

Definition

A quadruple (X, R, f, G) is a <u>transitive system</u> if:

- X is a nonempty set,
- R a reflexive and transitive relation on X,
- G is an abelian group and
- $f : R \to G$ is a <u>transitive function</u> on R, i.e.

$$(x, y), (y, z) \in R \Rightarrow f(x, y) + f(y, z) = f(x, z).$$

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In other words: these are graphs that are

- directed,
- weighted,
- transitive,
- with all vertices looped; simple otherwise,
- with triangle equality of the weights.

They appear in:

- Conservative force fields
- Financial tran$action$
- Linear algebra
- Combinatorics and graph theory
- Decision theory
  - Comparison consistent matrices in Analytic Hierarchy Process (T. Saaty 1977) satisfying

$$a_{ij}a_{jk} = a_{ik}.$$

- etc. etc. etc.

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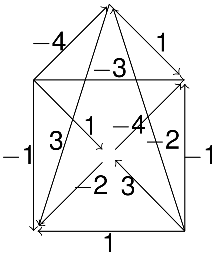

Figure: A transitive system that can be completed in $\mathbb{Z}$.

Basic completion theorems

Let $R' = \{(y, x) : (x, y) \in R\}$ and $\bar{f} : R \cup R' \to G$;

$$\bar{f}(x, y) = \begin{cases} f(x, y) & \text{if} \quad (x, y) \in R \\ -f(y, x) & \text{if} \quad (x, y) \in R' \end{cases}$$

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A transitive system can be completed if and only if

- *For every cycle $C = (x_1, x_2, \ldots, x_n = x_1)$ with all $(x_i, x_{i+1}) \in R \cup R'$, the circulation*

$$\sigma_G(C) = \sum_{i=1}^{n-1} \bar{f}(x_i, x_{i+1}) = 0.$$

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- *There is a labeling $\phi : X \to G$ such that*

$$f(x, y) = \phi(x) - \phi(y).$$

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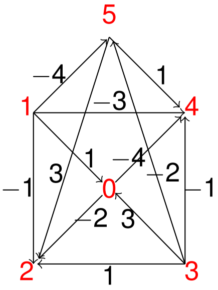

Figure: *The labeling is possible.*

Systems that cannot be completed

Theorem

If a transitive system (X, R, f, G) *cannot be completed then the graph of* R *contains an* S-cycle*, i.e. a* $2k$*-cycle (*$k > 2$*) with its consecutive edges pointing in opposite directions and a non-zero circulation.*

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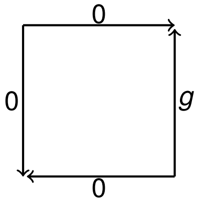

Figure: A transitive system that cannot be completed if $g \neq 0$.

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- *The system with G replaced by $\mathbb{Z}_{p^k}$, for some prime p.*

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Conjecture: every transitive graph is conclusive.

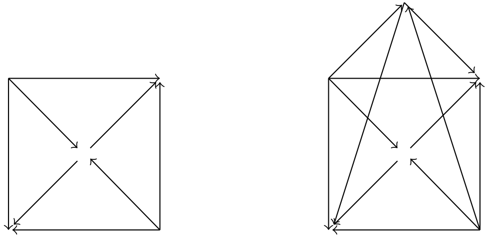

Figure: Soluble graphs.

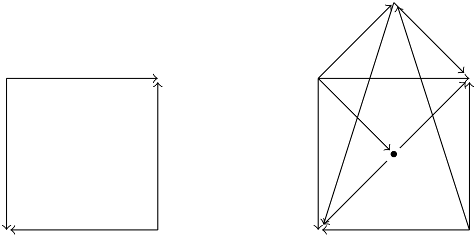

Figure: Defective graphs.

Algebraic approach

$\mathbb{R}$-solubility can be established by means of checking consistency of a finite number of systems of linear equations determined by the graph alone. There exists a feasible algorithm to determine solubility over torsion-free abelian groups.

Properties of soluble posets

Let X and Y be posets (so the graphs have the edges directed only one way).

- If $X \cap Y = \emptyset$ and at least one of X and Y is connected then their sum $X \oplus Y$ (Y is put on top of X) is soluble. If both A and B are disconnected, then $A \oplus B$ is defective.

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- If X and Y are soluble and Y is connected then the lexicographic product $X \overrightarrow{\times} Y$ is soluble.

Bipartite graphs

All bipartite graphs, as Hasse diagrams of posets, are conclusive.

Example

Any complete bipartite graph $K_{m,n}$ with $m, n > 1$ is defective since it is a sum of two disconnected posets.

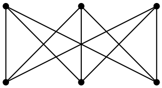

Figure: $K_{3,3}$ *is defective.*

Extension methods

Let $(X, \mathcal{R})$ be a poset and $p \in X$. Define:

- $X_p = X \setminus p$, $\mathcal{R}_p = \mathcal{R} \cap (X_p \times X_p)$,
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- A vertex p is an extension vertex if X^p is soluble within X_p.

Theorem

- *If p is an extension vertex and X_p is defective, then X is defective.*

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- *If every component of X_p contains at most one component of X^p and X_p is soluble, then X is soluble.*
- *If p is an extension vertex and X_p is conclusive, then X is conclusive.*

Examples

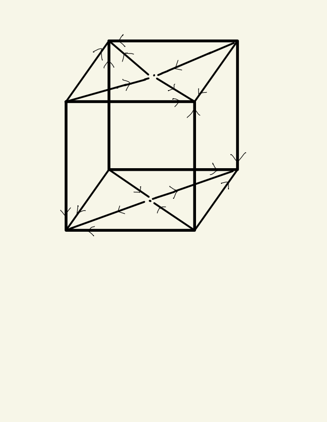


Figure: A given poset.

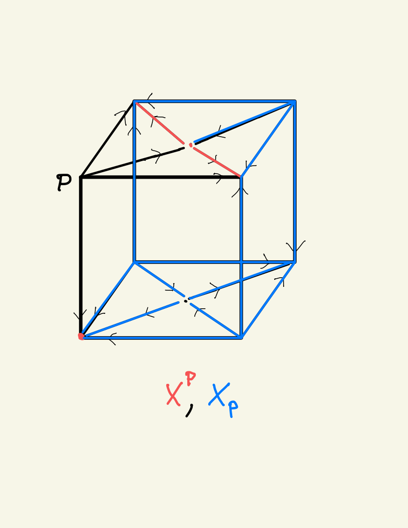
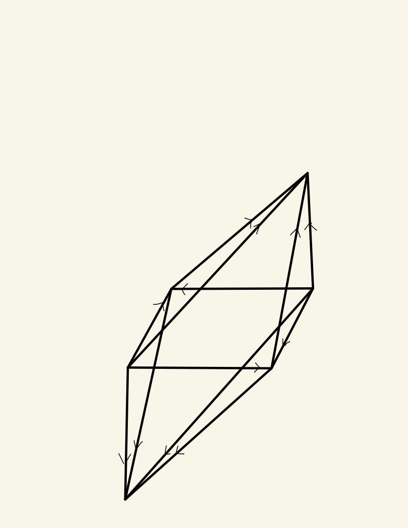


Figure: X^p is disconnected while X_p is connected, therefore X is defective.



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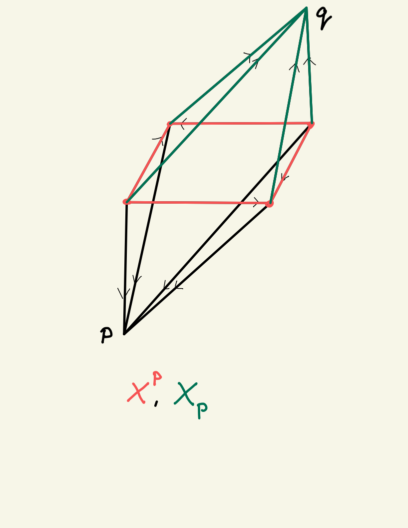

Figure: X^p is connected and X_p is soluble, therefore X is soluble.

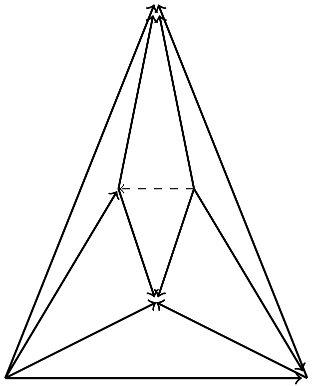

Figure: The octahedron - plane graph, soluble.

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- If X has at most 2 sinks or 2 sources, then the graph is conclusive.

Significance of the conjecture

Recall the conjecture: all transitive graphs are conclusive.

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If the conjecture is true, verification if a transitive graph is soluble or defective can be done by using only the group $\mathbb{Z}_2$, so the process would be minimized.

Transitive systems

Soluble, defective, conclusive graphs

Optimization of the verification of solubility

Possible applications

Latest results and current work

References

Towards an application in Analytic Hierarchy Process

Partially consistent comparison matrix configurations:

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$$\begin{bmatrix} 1 & & \bullet & & & \\ \bullet & 1 & \bullet & \bullet & \bullet & \\ & & 1 & & & \\ & & \bullet & 1 & \bullet & \\ & & & & 1 & \\ \bullet & & \bullet & \bullet & \bullet & 1 \end{bmatrix}$$

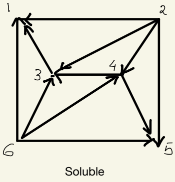
Can always be completed to a consistent matrix

$$\begin{bmatrix} 1 & & \bullet & & & \\ \bullet & 1 & \bullet & \bullet & \bullet & \\ & & 1 & & & \\ & & \bullet & 1 & \bullet & \\ & & & & 1 & \\ \bullet & & \bullet & & \bullet & 1 \end{bmatrix}$$

May not be possible to complete

Why?...

... Because:



Bounds on the degree

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- Otherwise for every $p \in X$, X^p is defective. In particular, all vertices have degree 4.
- It follows that X is an octahedron and thus it is soluble.

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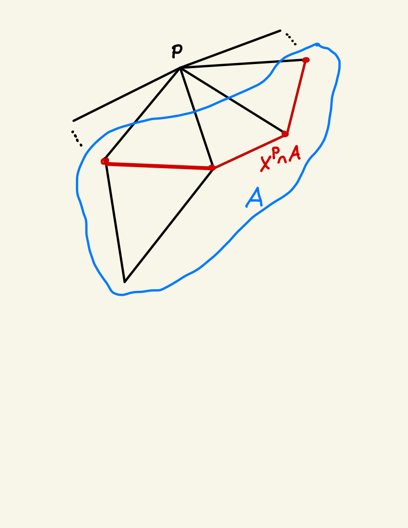

Figure: $X^p \cap A$ must be connected.

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- Hence, by the Extension Theorem X is soluble.

Towards conclusiveness of planar graphs

Suppose A is a mid-vertex. Unless A is an extension vertex, X can be drawn this way:

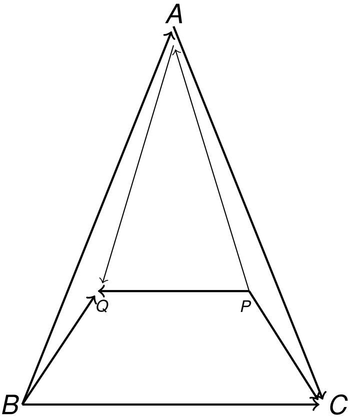

Figure: In-deg(A) $=$ Out-deg(A) $= 2$, B, P - sources, C, Q -sinks.

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- Otherwise we build a transitive system on X that cannot be completed:

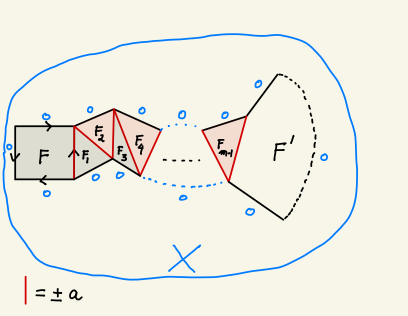
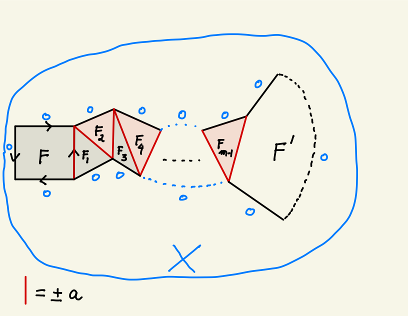


Figure: A triangular path from F to F'.



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- A weight of $\pm a \in G$ is assigned along the edges of the adjacent triangular faces, shown in red, and $0's$ to all remaining edges.

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Theorem (99%)

Every planar graph is conclusive.

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THANK YOU FOR YOUR ATTENTION