Cover-free families on Graphs

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Overview

Cover-free families: Definition

Definition

A set system is d-cover-free if no set is covered by the union of d others.

Example: 2-cover-free family

Set system:

$$\mathcal{X} = \{1,2,3,4,5,6,7,8,9\}$$
$$\mathcal{B}$$

$$B_a = \{1,2,3\}$$
$$B_b = \{4,5,6\}$$
$$B_c = \{7,8,9\}$$
$$B_d = \{1,4,7\}$$
$$B_e = \{2,5,8\}$$
$$B_f = \{3,6,9\}$$
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- $B_d = \{1,4,7\} \not\subseteq \{1,2,3,4,5,6\} = B_a \cup B_b$.
- $(\mathcal{X}, \mathcal{B})$ is not 3-cover-free.

Cover-free families: Representation using a binary matrix

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	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
9	0	0	1	0	0	1	1	0	0	0	1	0

Table: 2-CFF$(9, 12)$

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4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
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5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
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4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
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5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
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Cover-free families: Definitions

Definition

A set system is *d-cover-free* if no set is covered by the union of d others.

Definition (Alternate definition of cover-free families)

Given $d < t \leq n$ positive integers, a d-CFF(t, n) is a $t \times n$ binary matrix M such that any set of $d+1$ columns has a permutation sub-matrix of dimension $d+1$.

Minimizing number of rows

Definition (Minimizing number of Rows)

Given d and n we want to minimize the number of rows:

$$t(d, n) = \min\{t : \exists \text{ a } d\text{-CFF}(t, n)\}.$$

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The case for $d = 1$ is solved due to Sperner's theorem.
Given n, we have

$$t(1, n) = \min\{s : \binom{s}{\lfloor s/2 \rfloor} \geq n\}.$$

Example

$$t(1, 6) = 4.$$
$$\mathcal{X} = \{1, 2, 3, 4\}.$$
$$\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

Cover free families: minimizing number of rows

For $d \geq 2$,

$$c_1 \frac{d^2}{\log(d)} \log(n) \leq t(d,n) \leq c_2 d^2 \log(n).$$

Lower bounds: Dyachkov & Rikov (1982), Ruszinkó (1994), Füredi (1996).

Upper bounds: (Constructive)

Porat and Rothschild (2010): deterministic polynomial-time.

Gargano, Rescigno, Vaccaro (2020).

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Theorem (Erdös, Frankl, and Füredi (1982))

$$3.106 \log(n) < t(2,n) < 5.512 \log(n).$$

2-CFF(9, 12) on a graph

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
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4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
9	0	0	1	0	0	1	1	0	0	0	1	0

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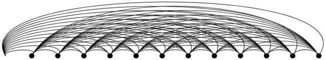

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1	1	0	0	1	0	0	1	0	0	1	0	0
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4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
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Cover-free families on Graphs

Definition

A *cover-free family on a graph G*, denoted $\overline{G}$-CFF, is a set system such that, for each edge in G,

- Their union does not contain any other subset in the system.
- The corresponding pair of subsets are not contained in one another.

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- **Notation:** $t(G) = \min\{t : \exists \text{ a } \overline{G}\text{-CFF}(t, |V(G)|\}$.
- If $G = K_n$, then $t(G) = t(2, n)$.

2-CFF(9, 12) on a graph

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
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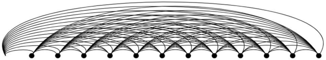

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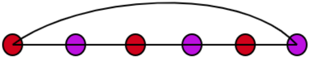
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
9	0	0	1	0	0	1	1	0	0	0	1	0

Table: 2-CFF(9, 12)

$$2\text{-CFF}(9,12) = \overline{K_{12}}\text{-CFF}.$$

$t(2,12) = 9 = t(K_{12})$. (Lee, VanRees, Wei (2006))

A $\overline{C_6}$-CFF



1	2	3	4	5	6
1	0	0	0	0	1
0	1	0	0	1	0
0	0	1	1	0	0
1	1	1	0	0	0
0	0	0	1	1	1

Figure: A $\overline{C_6}$-CFF

$\overline{C_6}$-CFF as the set system:

$$\mathcal{X} = \{1, 2, 3, 4, 5\}$$

$$[\{1, 4\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{1, 5\}].$$

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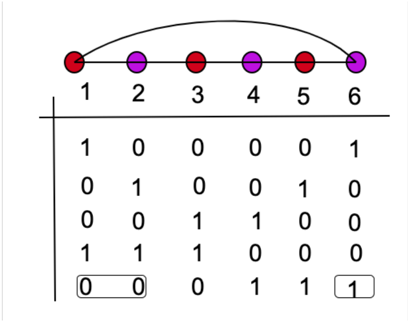

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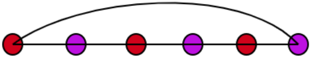

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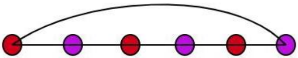

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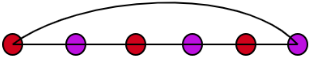

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- Their union does not contain any other subset in the system. G-CFF

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Remarks: We denote by $t_e(G)$ and $t_{in}(G)$ the minimum t such that there exist a G-CFF and G-in-CFF; respectively.

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Proposition (Idalino, Moura (2025+))

$$t(G) \leq t_e(G) + t_{in}(G).$$

Some bounds of $\overline{G}$-CFF

Theorem (Idalino, Moura (2025+))

$$t(G) \leq \chi(G) \log(n)$$

where $\chi(G)$ is the Chromatic number of G.

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Corollary

Let P_n and C_n be a Path and a Cycle of length n, respectively.

$$t(P_n) \leq 2\log(n) \text{ and } t(C_n) \leq 2\log(n) + (n \mod 2).$$

Some bounds of $\overline{G}$-CFF

Theorem (P., Moura (2025+))

Let G be a connected graph with n vertices. Then,

$$t(1, n) \leq t(G) \leq t(2, n).$$

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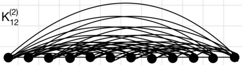
$K_{12}^{(1)}$

1	1	1	1	1	1	1	0	0	0	0	0	0
2	1	1	1	0	0	0	1	1	1	0	0	0
3	1	0	0	1	0	0	1	0	0	1	1	1
4	0	1	0	1	1	1	1	1	1	1	1	0
5	0	0	1	0	1	0	0	1	0	1	0	1
6	0	0	0	0	0	1	0	0	1	0	1	1

C_{12}

1	1	0	0	0	0	1	1	0	0	0	0	1
2	0	1	0	0	1	0	0	1	0	0	1	0
3	0	0	1	1	0	0	0	0	1	1	0	0
4	0	0	0	1	1	1	1	1	1	0	0	0
5	1	1	1	0	0	0	0	0	0	1	1	1
6	0	0	0	0	0	0	1	1	1	1	1	1
7	1	1	1	1	1	1	0	0	0	0	0	0

$K_{12}^{(2)}$



1	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
9	0	0	1	0	0	1	1	0	0	0	1	0

$$t(K_{12}^{(1)}) = 6, t(C_{12}) \leq 7, \text{ and } t(K_{12}) = 9$$

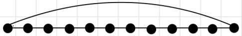
Some bounds of $\overline{G}$-CFF

Theorem (P., Moura (2025+))

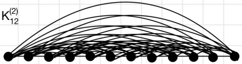
Let G be a connected graph with n vertices. Then,

$$t(1, n) \le t(G) \le t(2, n).$$

1	1	1	1	1	1	1	0	0	0	0	0	0
2	1	1	1	0	0	0	1	1	1	0	0	0
3	1	0	0	1	0	0	1	0	0	1	1	1
4	0	1	0	1	1	1	1	1	1	1	1	0
5	0	0	1	0	1	0	0	1	0	1	0	1
6	0	0	0	0	0	1	0	0	1	0	1	1



1	1	0	0	0	0	1	1	0	0	0	0	1
2	0	1	0	0	1	0	0	1	0	0	1	0
3	0	0	1	1	0	0	0	0	1	1	0	0
4	0	0	0	1	1	1	1	1	1	0	0	0
5	1	1	1	0	0	0	0	0	0	1	1	1
6	0	0	0	0	0	0	1	1	1	1	1	1
7	1	1	1	1	1	1	0	0	0	0	0	0

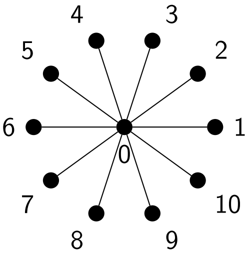


1	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0	0	0	1
6	0	1	0	0	0	1	0	1	0	1	0	0
7	0	0	1	1	0	0	0	1	0	0	0	1
8	0	0	1	0	1	0	0	0	1	1	0	0
9	0	0	1	0	0	1	1	0	0	0	1	0

$t(K_{12}^{(1)}) = 6$, $t(C_{12}) \le 7$, and $t(K_{12}) = 9$

$6 \le t(C_{12}) \le 7$

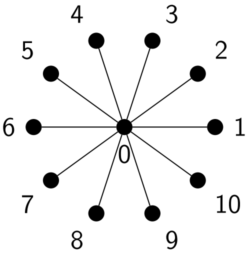
Cover-free family on a Star Graph



	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	0	0	0	0	0	0
2	0	1	0	0	0	1	1	1	0	0	0
3	0	0	1	0	0	1	0	0	1	1	0
4	0	0	0	1	0	0	1	0	1	0	1
5	0	0	0	0	1	0	0	1	0	1	1

Figure: A star graph S_{11} (left) and a S_{11}-CFF with $t_e(S_{11}) = 5$.

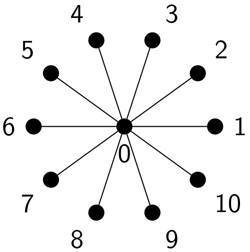
Cover-free family on a Star Graph



	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	1	1	1	1	1	1
2	1	0	0	0	0	0	0	0	0	0	0

Figure: A star graph S_{11} (left) and a S_{11}-in-CFF with $t_{in}(S_{11}) = 2$.

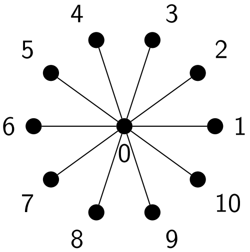
Cover-free family on a Star Graph



	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	0	0	0	0	0	0
2	0	1	0	0	0	1	1	1	0	0	0
3	0	0	1	0	0	1	0	0	1	1	0
4	0	0	0	1	0	0	1	0	1	0	1
5	0	0	0	0	1	0	0	1	0	1	1
6	0	1	1	1	1	1	1	1	1	1	1
7	1	0	0	0	0	0	0	0	0	0	0

Figure: A star graph S_{11} (left) and a $\overline{S_{11}}$-CFF with $t(S_{11}) \leq 7$.

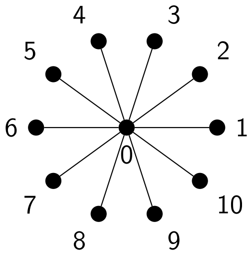
Cover-free family on a Star Graph



	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	0	0	0	0	0	0
2	0	1	0	0	0	1	1	1	0	0	0
3	0	0	1	0	0	1	0	0	1	1	0
4	0	0	0	1	0	0	1	0	1	0	1
5	0	0	0	0	1	0	0	1	0	1	1
6	1	0	0	0	0	0	0	0	0	0	0

Figure: A star graph S_{11} (left) and a $\overline{S_{11}}$-CFF with $t(S_{11}) = 6$.

Cover-free family on a Star Graph



	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	0	0	0	0	0	0
2	0	1	0	0	0	1	1	1	0	0	0
3	0	0	1	0	0	1	0	0	1	1	0
4	0	0	0	1	0	0	1	0	1	0	1
5	0	0	0	0	1	0	0	1	0	1	1
6	1	0	0	0	0	0	0	0	0	0	0

Figure: A star graph S_{11} (left) and a $\overline{S_{11}}$-CFF with $t(S_{11}) = 6$.

Summary of the main results

1. There is an infinite family of Star graphs S_n such that $t(S_n) = t(1, n)$.

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2. $t_{in}(G) = t(1, \chi(G))$.

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1. There is an infinite family of Star graphs S_n such that $t(S_n) = t(1, n)$.
2. $t_{in}(G) = t(1, \chi(G))$.
3. $\log(n) \leq t(G) \leq 1.89 \log(n)$ when G is either P_n or C_n (Construction using a Mixed-Radix Gray Code).

An infinite family meeting the lower bound $t(1, n) \leq t(G)$

Theorem (P., Moura, 2025+)

Let $G = S_n$ be a star graph on n vertices. Then, $t(S_n) = t(1, n-1) + 1$.

Corollary (P., Moura, 2025+)

If $n = \binom{x}{\lfloor x/2 \rfloor} + 1$ for any $x \in \mathbb{N}$, then $t(S_n) = t(1, n)$, thus meeting the lower bound $t(1, n) \leq t(G)$.

Proof.

$t(S_n) = t(1, \binom{x}{\lfloor x/2 \rfloor}) + 1 = x + 1 = t(1, n)$. □

$t_{in}(G) = t(1, \chi(G))$.

Characterizing t_{in} for Graphs via homomorphisms to Sperner Graphs:

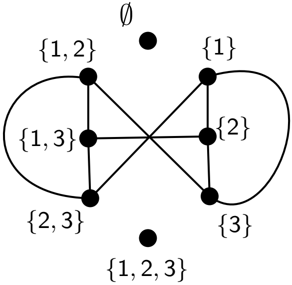


Figure: Sperner graph of order 3

$t_{in}(G) = t(1, \chi(G))$.

Sketch of the proof:

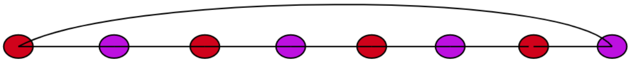
We first prove some basic results using Graph homomorphism:

- If $G \to H$, then $t_{in}(G) \leq t_{in}(H)$.
- $t_{in}(K_n) = t(1, n)$.
- $\omega(\mathcal{S}(z)) = \chi(\mathcal{S}(z)) = \binom{z}{\lfloor \frac{z}{2} \rfloor}$. (Using Dilworth's theorem[1])
- $t_{in}(\mathcal{S}(z)) = z$.
- $t_{in}(G) = \min_{l \in \mathbb{N}}\{l : G \to \mathcal{S}(l)\}$.

Since $G \to K_{\chi(G)}$, $t_{in}(G) \leq t_{in}(K_{\chi(G)}) = t(1, \chi(G))$.
Now, suppose by contradicton, $t_{in}(G) < t(1, \chi(G)) = k$. So, $\min\{l : G \to \mathcal{S}(l)\} = t_{in}(G) \leq k - 1$. Thus, $G \to \mathcal{S}(k-1)$.
However, $\chi(G) \leq \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}$, which follows that $t(1, \chi(G)) \leq k - 1$, the desired contradiction. □

[1] R P Dilworth, A Decomposition Theorem for Partially Ordered Sets, Annals of Mathematics (1950)

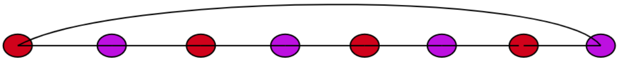
Cover-free families on Paths and Cycles



	1	2	3	4	5	6	7	8
1	1	1	1	1	0	0	0	0
2	0	0	0	0	1	1	1	1
3	1	1	0	0	0	0	1	1
4	0	0	1	1	1	1	0	0
5	1	0	0	1	1	0	0	1
6	0	1	1	0	0	1	1	0

Figure: A $\overline{C_8}$-CFF with $t(C_8) = 6$.

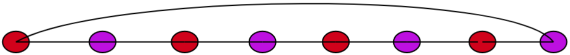
Cover-free families on Paths and Cycles



	1	2	3	4	5	6	7	8
1	1	1	1	1	0	0	0	0
2	0	0	0	0	1	1	1	1
3	1	1	0	0	0	0	1	1
4	0	0	1	1	1	1	0	0
5	1	0	0	1	1	0	0	1
6	0	1	1	0	0	1	1	0

Figure: A $\overline{C_8}$-CFF with $t(C_8) = 6$.

Cover-free families on Paths and Cycles



	1	2	3	4	5	6	7	8
1	1	1	1	1	0	0	0	0
2	0	0	0	0	1	1	1	1
3	1	1	0	0	0	0	1	1
4	0	0	1	1	1	1	0	0
5	1	0	0	1	1	0	0	1
6	0	1	1	0	0	1	1	0
	{1,3,5}	{1,3,6}	{1,4,6}	{1,4,5}	{2,4,5}	{2,4,6}	{2,3,6}	{2,3,5}

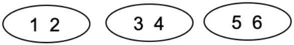
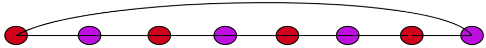


Figure: A $\overline{C_8}$-CFF with $t(C_8) = 6$.

Cover-free families on Paths and Cycles



	1	2	3	4	5	6	7	8
1	1	1	1	1	0	0	0	0
2	0	0	0	0	1	1	1	1
3	1	1	0	0	0	0	1	1
4	0	0	1	1	1	1	0	0
5	1	0	0	1	1	0	0	1
6	0	1	1	0	0	1	1	0
	{1,3,5}	{1,3,6}	{1,4,6}	{1,4,5}	{2,4,5}	{2,4,6}	{2,3,6}	{2,3,5}
	000	001	011	010	110	111	101	100

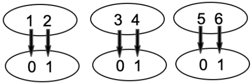


Figure: A $\overline{C_8}$-CFF with $t(C_8) = 6$ and the corresponding Binary Reflected Gray Code.

Cover-free families on Paths and Cycles

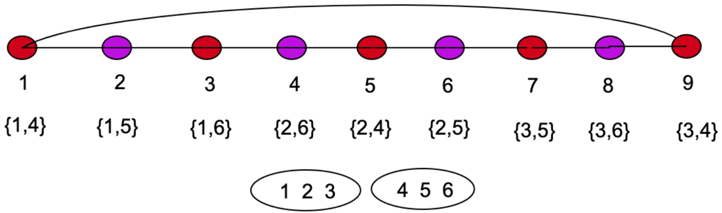


Figure: A $\overline{C_9}$-CFF with $t(C_9) = 6$.

Cover-free families on Paths and Cycles

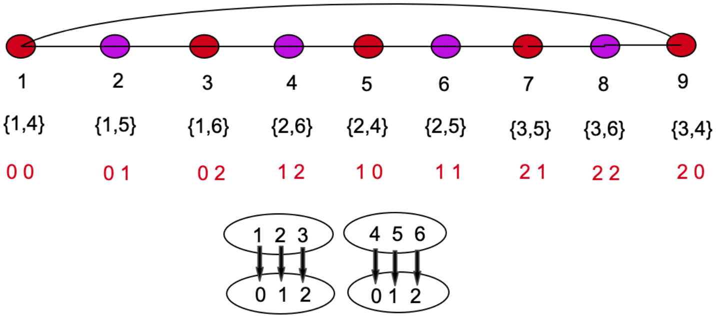


Figure: A $\overline{C_9}$-CFF with $t(C_9) = 6$ and the corresponding mixed-radix[2] Gray code.

[2] D.E. Knuth, The Art of Computer Programming, Volume 4A: Combinatorial Algorithms,

Optimal Integer Partitions for Product

Theorem (Sequence $A000792$ in Sloane's On-Line Encyclopedia of Integer Sequences)

Let $a(m)$ be the function which gives the maximum product of size of partitions of $[m]$. Then,

$$a(m) = \begin{cases} 3^k & \text{if } m = 3k, \\ 4 \cdot 3^{k-1} & \text{if } m = 3k+1, \\ 2 \cdot 3^k & \text{if } m = 3k+2. \end{cases}$$

Bounds of $\overline{P_n}$-CFF and $\overline{C_n}$-CFF

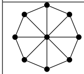
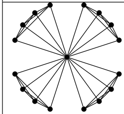
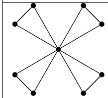
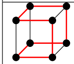
Theorem (P., Moura (2025+))

For some $k \geq 1$,

$$t(G) \leq \begin{cases} 3k & \text{if } n \in (2 \cdot 3^{k-1}, 3^k] \\ 3k+1 & \text{if } n \in (3^k, 4 \cdot 3^{k-1}] \\ 3k+2 & \text{if } n \in (4 \cdot 3^{k-1}, 2 \cdot 3^k] \end{cases}$$

where G is either P_n or C_n.
For all the above cases, $t(G) \leq \frac{3}{\log_2(3)} \log_2(n) + o(1)$ where $\frac{3}{\log_2(3)} = 1.8915\cdots$.

Bounds of CFFs on other families of graphs:

Graph Type	$t(G)$
 Wheel graph	$t(C_{n+1}) \leq t(W_{n+1}) \leq t(C_n) + 1.$
 Windmill graph	$t(1,(k-1)n) + 1 \leq t(Wd(k,n)) \leq t(1,n) + t(2,k-1) + 1.$
 Friendship graph	If $n \in \left[\binom{2k-1}{k}+1, \left\lfloor \frac{1}{2}\binom{2k+1}{k} \right\rfloor\right]$, then $t(1,n)+2 \leq t(F_{2n+1}) \leq t(1,n)+3.$ If $n \in \left[\left\lfloor \frac{1}{2}\binom{2k+1}{k} \right\rfloor + 1, \binom{2k}{k}\right]$, then $t(F_{2n+1}) = t(1,n)+3.$
 Hypercube graph	$t(C_{2^n}) \leq t(Q_n) \leq 2n.$

Future work

- We are investigating tight bounds for $t(G)$ for specific classes of graphs.
- Cover-free families on hypergraphs and the product of hypergraphs (**Work in progress**).
- Further develop the theory, constructions and bounds for CFFs on hypergraphs.
- Non-existential results of cover-free families on hypergraphs using the probabilistic method.
- Generalization of cover-free families on hypergraphs.

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The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1.
Pearson Education India, 2011.

Erdős, P., Frankl, P., and Füredi, Z.
Families of finite sets in which no set is covered by the union of two others.
J. Comb. Theory, Series A, 33(2):158–166, 1982.

Idalino, T.B., and Moura, L.
Group testing and Cover-free families on Hypergraphs.
Manuscript to be submitted, 2025.

Sperner, E.
Ein Satz über Untermengen einer endlichen Menge.
Mathematische Zeitschrift, 27(1):544–548, 1928.

E. Porat and A. Rothschild.
Explicit nonadaptive combinatorial group testing schemes.
IEEE Transactions on Information Theory, 57(12):7982–7989, 2011.

P. C. Li, G. H. J. Van Rees, and R. Wei.
Constructions of 2-cover-free families and related separating hash families.
Journal of Combinatorial Designs, 14(6):423–440, 2006.

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Thank you for your attention! :)