Integral Hypergraphs

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Introduction

In Spectral Graph Theory, matrices are associated to graphs and graph structure is studied through eigenvalues and eigenvectors of these matrices.

Although the study of hypergraphs and their structural properties can be considered a fruitful area, with many published articles, a Spectral Theory for hypergraphs is still at an early stage.

We will deal here with a classic problem of Spectral Graph Theory, in the context of hypergraphs.

Integral Graphs

- A graph is integral if the spectrum of its adjacency matrix consists entirely of integers.
- Question posed by Harary and Schwenk, 74:
  Which graphs have integral spectra?
- Some examples: the complete graph K_n, the complete bipartite graph $K_{p,q}$ where $p \cdot q = t^2$, the path P_2 and the cycles C_3, C_4 and C_6.
- K. Balińska *et al.* (2002) noticed in their survey on integral graphs that they are very rare and difficult to find.

Although the initial motivation was strictly theoretical, it was later seen that integral graphs were important in certain applications:

M. Christandl, N. Datta, A. Ekert, A.J. Landahl, Perfect state transfer in quantum spin networks, Phys. Rev. Lett. 92 (2004) 187.

Hypergraph

There are two distinct approaches to a spectral theory of hypergraphs:

- through matrices
- through tensors.

Here we will approach the study of hypergraphs through the **adjacency matrix** and see how relevant problems in spectral graph theory can be extended to hypergraphs.

Preliminaries

In this section, basic notions about hypergraphs and the adjacency matrix are presented.

Hypergraphs

- A *hypergraph* $\mathcal{H} = (V, E)$ is given by a finite vertex set V and an edge set $E = \{e : e \subseteq V\}$, where $|e| \geq 2$.
- $\mathcal{H}$ is said to be k-uniform if $|e| = k$, $\forall e \in E$.

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Remark

A graph is a $2-$*uniform hypergraph.*

Example

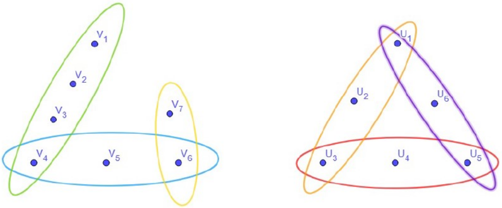


Figure: A non uniform hypergraph and a 3−uniform hypergraph

Adjacency matrix of a hypergraph

Definition: Let $\mathcal{H}$ be a hypergraph with n vertices.

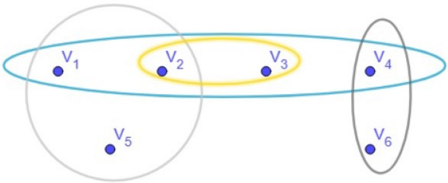
The *adjacency matrix* of $\mathcal{H}$, $\mathbf{A}(\mathcal{H})$, is the $n \times n$ symmetric matrix with entries:

$$a_{ij} = |\{e \in E(\mathcal{H}) : v_i, v_j \in e\}|.$$

The eigenvalues of $\mathbf{A}(\mathcal{H})$ are $\lambda_1(\mathcal{H}) \geq \ldots \geq \lambda_n(\mathcal{H})$.

The *spectrum* of a hypergraph $\mathcal{H}$, $spec(\mathcal{H})$, is the spectrum of its adjacency matrix.

Example



$$\mathbf{A}(\mathcal{H}) = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \textit{and}$$

$$spec(\mathcal{H}) = \{(4,41)^1, (0,78)^1, (0,06)^1, (-1,23)^1, (-1,59)^1, (-2,43)^1\}.$$

Remark

Note that the adjacency matrix of a hypergraph is real and symmetric. Therefore:

- (i) *It is diagonalizable with real eigenvalues.*
- (ii) *As the trace is zero, it always has positive and negative eigenvalues, unless it is the hypergraph without edges.*

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A natural question is if it is also possible to identify the integrality of hypercycles.

Try to answer this question will be the focus of this part of the presentation

Hypercycles

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Definition: Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the ring of integers modulo n. A *$k-$uniform hypercycle* on n vertices, $k < n$, $C_n^{[k]}$, is given by a vertex set $V = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and an edge set $E = \{e_j \mid j \in V\}$ where $e_j = \{j, j+1, \ldots, j+k-1\}$, $\forall j \in V$.

As in 2-uniform cycles, the adjacency matrix of a k-uniform hypercycle is also a circulant matrix.

Definition: A *circulant matrix* is a square matrix in which all row are composed of the same elements and each row is rotated one element to the right relative to the previous row.

Example

$C_8^{[4]}$ is given by the sets:
$V = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$E = \{\{0, 1, 2, 3\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5, 6\}, \{4, 5, 6, 7\}, \{5, 6, 7, 0\}, \{6, 7, 0, 1\}, \{7, 0, 1, 2\}\}.$

Its adjacency matrix is:

$$\begin{bmatrix} 0 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 0 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 0 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 0 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

Recall that a circulant matrix is fully characterized by the first row. If $[0, a_1, a_2, \ldots, a_{n-1}]$ is the first row, its eigenvalues are given by

$$\lambda_j = \sum_{k=1}^{n-1} a_k \omega^{kj}, \quad \text{for} \quad j = 0, \ldots, n-1,$$

where $\omega = exp(\frac{2\pi i}{n}) = cos(\frac{2\pi}{n}) + i.sin(\frac{2\pi}{n})$ and i is the imaginary unit.

Remark

For $k = n$ we have that $\mathbf{A}(C_n^{[n]}) = J - \mathbf{I}$.

For $k = n - 1$ we have that $\mathbf{A}(C_n^{[n-1]}) = (n-2)(J - \mathbf{I})$.

Therefore, $C_n^{[n]}$ and $C_n^{[n-1]}$ are integral hypercycles, for every $n \geq 3$.

We are looking for integral hypercycles $C_n^{[k]}$, for $k \leq n - 2$.

3-uniform hypercycles

Next we study the eigenvalues of the adjacency matrix of $C_n^{[3]}$.
For $n \geq 5$ we have by computation that:

- $\lambda_0 = 6$.
- $\lambda_j = 4cos\left(\frac{2\pi j}{n}\right) + 2cos\left(\frac{4\pi j}{n}\right)$, for $1 \leq j \leq n-1$.

Proposition

If $n \geq 5$ the spectrum of $C_n^{[3]}$ is given by:

1. *For n odd,*
$$\left\{(6)^1, \left(4cos\left(\frac{2\pi}{n}\right) + 2cos\left(\frac{4\pi}{n}\right)\right)^2, \left(4cos\left(\frac{4\pi}{n}\right) + 2cos\left(\frac{8\pi}{n}\right)\right)^2, \ldots, \left(4cos\left(\frac{(n-1)\pi}{n}\right) + 2cos\left(\frac{2(n-1)\pi}{n}\right)\right)^2\right\}.$$

2. *For n even,*
$$\left\{(6)^1, \left(4cos\left(\frac{2\pi}{n}\right) + 2cos\left(\frac{4\pi}{n}\right)\right)^2, \left(4cos\left(\frac{4\pi}{n}\right) + 2cos\left(\frac{8\pi}{n}\right)\right)^2, \ldots, \left(4cos\left(\frac{(n-2)\pi}{n}\right) + 2cos\left(\frac{2(n-2)\pi}{n}\right)\right)^2, (-2)^1\right\}.$$

Proof: As seen before, we have that:

$$\lambda_0 = 6 \text{ and } \lambda_j = 4cos\left(\frac{2\pi j}{n}\right)+2cos\left(\frac{4\pi j}{n}\right), \text{ for } j = 1, \ldots, n-1.$$

But these numbers are not all distinct, since:
$cos\left(\frac{2\pi j}{n}\right) = cos\left(\frac{2\pi(n-j)}{n}\right)$ and $cos\left(\frac{4\pi j}{n}\right) = cos\left(\frac{4\pi(n-j)}{n}\right)$.

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Therefore, for n odd we have:
$\lambda_1 = \lambda_{n-1}, \lambda_2 = \lambda_{n-2}, \ldots, \lambda_{\frac{n-1}{2}} = \lambda_{\frac{n+1}{2}}$;

and for n even: $\lambda_1 = \lambda_{n-1}, \lambda_2 = \lambda_{n-2}, \ldots, \lambda_{\frac{n-2}{2}} = \lambda_{\frac{n+2}{2}}$.

Besides that, when n is even and $j = \frac{n}{2}$, we have that:

$$4cos\left(\frac{2\pi}{n}.\frac{n}{2}\right)+2cos\left(\frac{4\pi}{n}.\frac{n}{2}\right) = 4cos\left(\pi\right)+2cos\left(2\pi\right) = -4+2 = -2.$$

Since $\frac{n}{2} = n - \frac{n}{2}$, the multiplicity of this eigenvalue is one.

□

Theorem

Let $n \geq 5$. $C_n^{[3]}$ *is integral* $\iff$ $n = 6$.

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Proof:

- $\lambda_1(n) = 4cos\left(\frac{2\pi}{n}\right) + 2cos\left(\frac{4\pi}{n}\right)$ is an eigenvalue of $C_n^{[3]}$.
- $\lambda_1(n)$ is an increasing function for $n \geq 5$.
- $\lambda_1(n) \in (0, 6)$ for $n \geq 5$.
- $\lambda_1(16) > 5$, therefore $\lambda_1(n) \in (5, 6)$ and it is never an integer number for $n \geq 16$.
- For $5 \leq n \leq 15$, we computed that $C_n^{[3]}$ is integral $\iff$ $n = 6$. □

4-uniform hypercycles

Next we study the eigenvalues of the adjacency matrix of $C_n^{[4]}$.
For $n \geq 6$ we have that:

- $\lambda_0 = 12$.
- $\lambda_j = 6cos\left(\frac{2\pi j}{n}\right) + 4cos\left(\frac{4\pi j}{n}\right) + 2cos\left(\frac{6\pi j}{n}\right)$, for $1 \leq j \leq n-1$.

Then $\lambda_j < 12$, for $1 \leq j \leq n-1$.

Theorem

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We verified that for $n \geq 28$, $\lambda_1(n) > 11$, so it is not an integer as $\lambda_1(n) \in (11, 12)$.

For $6 \leq n \leq 27$, we computed that $C_n^{[4]}$ is integral $\iff n = 6$.

5-uniform hypercycles

Next we study the eigenvalues of the adjacency matrix of $C_n^{[5]}$.
For $n \geq 7$ we have that:

- $\lambda_0 = 20$.
- $\lambda_j = 8cos\left(\frac{2\pi j}{n}\right) + 6cos\left(\frac{4\pi j}{n}\right) + 4cos\left(\frac{6\pi j}{n}\right) + 2cos\left(\frac{8\pi j}{n}\right)$, for $1 \leq j \leq n-1$.

Then $\lambda_j < 20$, for $1 \leq j \leq n-1$.

Theorem

Let $n \geq 7$. $C_n^{[5]}$ is never an integral hypergraph.

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We verified that for $n \geq 45$, $\lambda_1(n) > 19$, so it is not an integer as $\lambda_1(n) \in (19, 20)$.

For $7 \leq n \leq 44$, we computed that $C_n^{[5]}$ is never integral.

Conclusion

Besides the cases where $n = k$ $n = k + 1$, it was proven that for $k = 3$ and $k = 4$, $C_n^{[k]}$ is integral $\iff$ $n = 6$. For $k = 5$ it was proven that $C_n^{[5]}$ is never integral. Based on that, this question was left open:

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- For $k \geq 6$ and $n > k + 1$, is there any other integral k-uniform hypercycle $C_n^{[k]}$?

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Besides the cases where $n = k$ $n = k + 1$, it was proven that for $k = 3$ and $k = 4$, $C_n^{[k]}$ is integral $\iff$ $n = 6$. For $k = 5$ it was proven that $C_n^{[5]}$ is never integral. Based on that, this question was left open:

- For $k \geq 6$ and $n > k + 1$, is there any other integral k-uniform hypercycle $C_n^{[k]}$?

We did not find any example. Excluding the case $n = k + 1$, only $C_6^{[3]}$ and $C_6^{[4]}$, together with the $2-$uniform cycles C_4 and C_6 are integral.

In conclusion, k-uniform integral hypercycles seem to be very difficult to find.

Integral hypergraphs G^k

We begin with the definition of one operation generating hypergraphs from a graph.

Definition: Let G be a graph and $k \geq 2$ an integer. The *k-power* G^k of G is a k-uniform hypergraph obtained from G by adding $k-2$ new vertices of degree one to each edge of G.

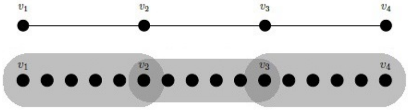

Figure: Graph $G = P_4$ an its respective G^6.

The adjacency matrix $A(G^k)$ is given in blocks by:

$$A(G^k) = \begin{bmatrix} A(G) & B(G) & B(G) & \dots & B(G) \\ B^T(G) & 0_m & I_m & \dots & I_m \\ B^T(G) & I_m & 0_m & \dots & I_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T(G) & I_m & I_m & \dots & 0_m \end{bmatrix},$$

Proposition: If G is an r-regular graph we have that

$$\lambda_1 = \frac{r+k-3+\sqrt{k^2-6k+6rk+r^2-10r+9}}{2},$$

$$\lambda_2 = \frac{r+k-3-\sqrt{k^2-6k+6rk+r^2-10r+9}}{2}$$

are eigenvalues of $A(G^k)$.

$\lambda_1 \in Z \Leftrightarrow p(r,k) = k^2 - 6k + 6rk + r^2 - 10r + 9$ is a perfect square.
Analysing computationally the Diophantine equation $p(r,k) = z^2$ we have:

r	**k**
2	none
3	7
4	5 or 16
5	10 or 29
6	4, 8, 17 or 46
7	7, 13, 26 or 67
8	11, 37 or 92
9	6, 16, 27, 50 or 121
10	9, 14, 22, 36, 65 or 154

Using this table, we looked for regular graphs G such that G^k is integral.

Example

Let $G = K_{6,6}$, then $(K_{6,6})^4$ is an integral hypergraph with 84 vertices, 36 edges.

$$Spect((K_{6,6})^4) = \{(9)^1, (4)^{10}, (1)^{25}, (-1)^{36}, (-2)^1, (-3)^{10}, (-6)^1\}.$$

Thanks!

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