Classification and Extremal Properties of Graphs Sharing Properties of Vertex-Transitive Graphs

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Let me start by giving credit where credit is due ...

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Definition

Let Γ be a k-**regular graph of girth** g; a (k, g)-graph.

- For each vertex v of Γ, let $\#g(v)$ be the number of girth-cycles containing v.
- For each edge e of Γ, let $\#g(e)$ be the number of girth-cycles containing e.

The **signature of a vertex** v of Γ, $sig(v)$, is the multiset $\{\#g(e) | e \text{ is incident to } v\}$.

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Note that

- If Γ is edge-transitive of girth g, $\#g(e) = \lambda$, for all $e \in E(\Gamma)$.
- If Γ is vertex-transitive of girth g, $\#g(v) = \Sigma$, for all $v \in V(\Gamma)$ and all the vertices have the same signature.

The Key Definitions

Definition

- A (k, g, λ)-**edge-girth-regular graph** Γ is a k-regular graph of girth g with the property that each edge of Γ is contained in λ g-cycles; $\#g(e) = \lambda$, for all $e \in E(\Gamma)$ (RJ, Kiss, Miklavič, 2018)

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- A $(k, g, \vec{a})$-**girth-regular graph** Γ **with signature** $\vec{a} = a_1 \leq a_2 \leq \ldots \leq a_k$ is a k-regular graph of girth g with the property that each vertex of Γ has signature $\vec{a}$, $sig(v) = \vec{a}$, for all $v \in V(\Gamma)$ (Potočnik, Vidali, 2019)

Motivation

- Moore graphs[1] are vertex-girth-regular, edge-girth-regular and girth-regular

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- vertex-transitive graphs are vertex-girth-regular and girth-regular
- edge-transitive graphs are edge-girth-regular, and if k-regular, then also girth-regular and vertex-girth-regular
- every k-regular edge-girth-regular graph is girth-regular with signatures consisting of equal values and vertex-girth-regular

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Some More Motivation ...

Theorem (Exoo, RJ, Mačaj, Širáň, 2019)

For any degree fixed degree $\Delta \geq 3$ and any positive integer δ, the order of the largest **vertex-transitive** *Δ-regular graph of diameter D differs from the Moore bound by more than δ for almost all diameters $D \geq 2$.*

Theorem (Filipovski, RJ, 2018)

For any fixed degree $k \geq 4$ and any positive integer δ, the order of the smallest **vertex-transitive** *k-regular graph of odd girth g differs from the Moore bound by more than δ for almost all odd girths $g \geq 3$.*

Analogues of the above theorems most likely hold for **vertex-girth-regular graphs** as well.

Basic Upper Bounds for Edge-Girth-Regular Graphs

For all edge-girth-regular graphs

$$0 < \lambda \leq (k-1)^{\frac{g-1}{2}}, \text{ for odd } g$$

$$0 < \lambda \leq (k-1)^{\frac{g}{2}}, \text{ for even } g$$

Census of Small Cubic Edge-Girth-Regular Graphs

- $k = 3$, $g = 3$
  - $\lambda = 2$, complete graph K_4, unique
- $k = 3$, $g = 4$
  - $\lambda = 4$, complete bipartite graph $K_{3,3}$, unique
  - $\lambda = 2$, cube, unique
- $k = 3$, $g = 5$
  - $\lambda = 4$, Petersen graph, unique
  - $\lambda = 2$, dodecahedron, unique
- $k = 3$, $g = 6$
  - $\lambda = 8$, Heawood graph, 14 vertices, unique
  - $\lambda = 6$, Möbius-Kantor graph, 16 vertices, unique
  - $\lambda = 4$, Pappus graph, 18 vertices, unique
  - $\lambda = 2$, **infinitely many graphs**
- $k = 3$, $g \geq 6$
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Census Based Questions

- Determine the triples (k, g, λ) for which there exists (or does not exist) an edge-girth-regular graph with these parameters

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Extremal vertex-girth-regular graphs are often (but not always) vertex-transitive and extremal edge-girth-regular graphs are often (but not always) edge-transitive

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Thus,

- finding parameter sets for which there exist graphs which are not vertex- or edge-transitive is **quite interesting**
- finding parameter sets for which there exist graphs which are not vertex- or edge-transitive and there are no vertex- or edge-transitive graphs with these parameters is **especially interesting**

Census of 4-regular edge-girth-regular graphs of girth 4

- $n_{egr}(4,4,9) = 8$ — $K_{4,4}$
- $n_{egr}(4,4,8) = \infty$ — **no graph exists,** Kiss, Miklavič, Szonyi
- $n_{egr}(4,4,7) = \infty$ — **no graph exists,** Kiss, Miklavič, Szonyi
- $n_{egr}(4,4,6) = 10$ — double-cover of K_5
- $n_{egr}(4,4,5) = 10$ — wreath-graph of length 5, $PX(5,1)$
- $n_{egr}(4,4,4) = \infty$ — **no graph exists,** Droogendijk 2024
- $n_{egr}(4,4,3) = 14$ — Goedgebeur, Jooken (2024)
- $n_{egr}(4,4,2) = 13$ — Goedgebeur, Jooken (2024)
- $n_{egr}(4,4,1) = 18$ — Goedgebeur, Jooken (2024)

And Symmetry Is Not Needed At All ...

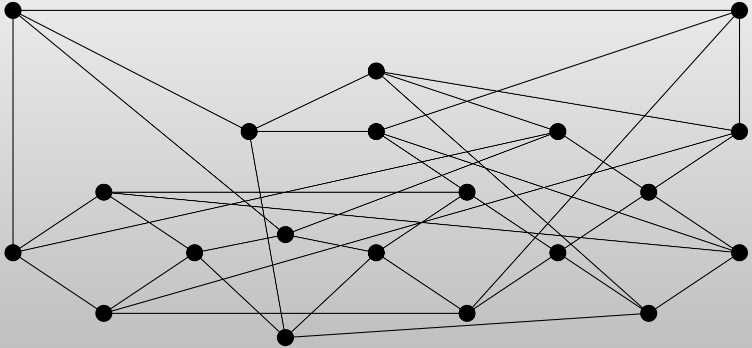
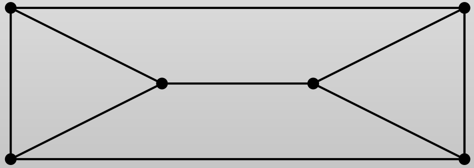


Figure: An asymmetric $(4, 4, 1)$-egr graph of order 20 found by Jooken

Census of Small Cubic Vertex-Girth-Regular Graphs

- $n_{vgr}(3,3,3) = 4$ K_4
- $n_{vgr}(3,3,2) = \infty$ **no graph exists**
- $n_{vgr}(3,3,1) = 6$



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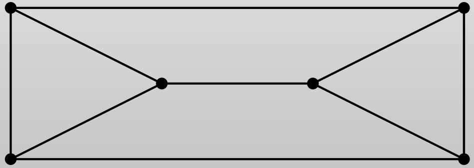

Lemma (Jooken, RJ, Porupsánszki, 2025+)

Let $k \geq 3$. There is no $(k, 3, \binom{k}{2} - 1)$-vgr graph.

Non-Existence Results for Vertex-Girth-Regular Graphs

Theorem (Jooken, RJ, Porupsánszki, 2025+)

Let $k \geq 3$, $g = 2s + 1 \geq 7$, *and* $0 < \epsilon \leq \frac{k-1}{2}$ *be integers. Then there is no* $(k, g, \max - \epsilon)$*-vgr graph.*

Theorem (Jooken, RJ, Porupsánszki, 2025+)

Let $k \geq 3$, $g = 2s \geq 4$, *and* $0 < \epsilon < k - 1$ *be integers. Then there is no* $(k, g, \max - \epsilon)$*-vgr graph.*

Finally ... Girth-Regular Graphs

Given integers k and g, for which tuples $(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k$ does there exist a girth-regular graph of girth g and signature $a_1 \leq a_2 \leq \ldots \leq a_k$?

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Theorem (Potočnik, Vidali, 2019)

If G is a girth-regular graph of valence k, girth g, and signature $(a_1, a_2, \ldots, a_k)$ then $a_k \le (k-1)^{\lfloor g/2 \rfloor}$.

Theorem (Potočnik, Vidali, 2019)

If G is a connected girth-regular graph of valence k, girth $2d$ for some integer d, and signature $(a_1, a_2, \ldots, a_k)$ such that $a_k = (k-1)^d$, then $a_1 = a_2 = \ldots = a_k$ and G is the incidence graph of a generalised d-gon of order $(k-1; k-1)$.
In particular, if $k = 3$, then $g \in \{4, 6, 8, 12\}$ g and G is isomorphic to $K_{3,3}$ (if $g = 4$), the Heawood graph (if $g = 6$), the Tutte-Coxeter graph (if $g = 8$) or to the Tutte 12-cage (if $g = 12$).

Girth-Regular Graphs vs. Vertex-Transitive Graphs

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- Classified **all cubic girth-regular graphs** of girth 5
- Classified **all cubic vertex-transitive graphs** of girth 6

Classification of Cubic Girth-Regular Graphs of Girth 6

Glevitzká, Jajcay, Lekse, Potočnik, 2025+

Theorem 1.2. *Let Γ be a cubic girth-regular graph of girth 6. Then the signature of Γ is one of the following:*

- *(a)* $(8,8,8)$ *with the only example being the Heawood graph, which is vertex-transitive,*
- *(b)* $(6,6,6)$ *with the only example being the Möbius-Kantor graph, which is vertex-transitive,*
- *(c)* $(4,5,5)$ *with the only example being the graph* $\Psi_9 \cong \Delta_3$*, which is vertex-transitive,*
- *(d)* $(4,4,4)$ *with only three non-isomorphic examples, two of which are vertex-transitive,*
- *(e)* $(3,4,5)$ *only for* Ψ_n *with* $n \geq 10$*, which are all vertex-transitive,*
- *(f)* $(2,3,3)$ *and there exist infinitely many of both vertex-transitive and non-vertex-transitive graphs, all of them being skeletons of maps of type* $\{6,3\}$ *on the torus or the Klein bottle,*
- *(g)* $(2,2,2)$ *and there exist infinitely many of both vertex-transitive and non-vertex-transitive graphs, all of them being skeletons of maps of type* $\{6,3\}$ *on the torus or the Klein bottle,*
- *(h)* $(1,1,2)$ *and there exist both vertex-transitive and non-vertex-transitive graphs, all of them being skeletons of truncations of some map with face cycles of length* 3*, or*
- *(i)* $(0,1,1)$ *and there exist both vertex-transitive and non-vertex-transitive graphs, all being truncations of a* 6*-regular graph (possibly with parallel edges).*

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… and we still have not found parameters for which there exists a non-vertex-transitive graph but no vertex-transitive

Thank you