# Design switching on graphs

Robin Simoens

Ghent University & Universitat Politècnica de Catalunya

3 June 2025 🍰

Based on joint work with Ferdinand Ihringer (SUSTech)

# Cospectral graphs

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$.

# Cospectral graphs

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$.

**Definition**

Graphs with the same spectrum are **cospectral**.

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

- Interesting for complexity theory

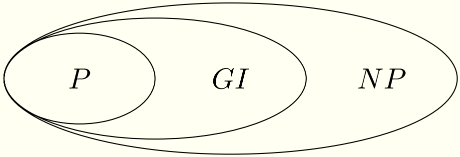



Figure: Is graph isomorphism an easy or hard problem?

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

- Interesting for complexity theory
- Interesting for chemistry

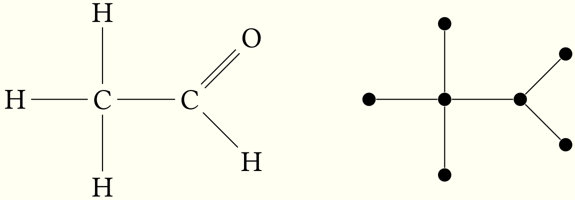



Figure: The molecular graph of acetaldehyde (ethanal).

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

- Interesting for complexity theory
- Interesting for chemistry

🙁 Almost all trees are **not** determined by their spectrum [Schwenk, 1973]

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

- Interesting for complexity theory
- Interesting for chemistry

🙁 Almost all trees are **not** determined by their spectrum [Schwenk, 1973]

🙁 Exponentially many graphs are **not** determined by their spectrum [Haemers and Spence, 2004]

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

- Interesting for complexity theory
- Interesting for chemistry

🙁 Almost all trees are **not** determined by their spectrum [Schwenk, 1973]

🙁 Exponentially many graphs are **not** determined by their spectrum [Haemers and Spence, 2004]

🙂 Computational evidence [Brouwer and Spence, 2009]

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| ratio | 1 | 1 | 0.941 | 0.936 | 0.895 | 0.861 | 0.814 | 0.787 | 0.789 |

# Cospectral graphs

**Conjecture (van Dam and Haemers, 2003)**

*Almost all graphs are determined by their spectrum.*

➤ Interesting for complexity theory
➤ Interesting for chemistry

🙁 Almost all trees are **not** determined by their spectrum [Schwenk, 1973]

🙁 Exponentially many graphs are **not** determined by their spectrum [Haemers and Spence, 2004]

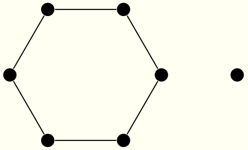
🙂 Computational evidence [Brouwer and Spence, 2009]

🙂 Exponentially many graphs are determined by their spectrum [Koval and Kwan, 2023]

# How to find cospectral graphs

**Theorem (Godsil and McKay, 1982)**

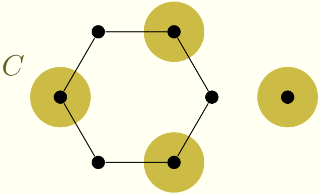
*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*



# How to find cospectral graphs

**Theorem (Godsil and McKay, 1982)**

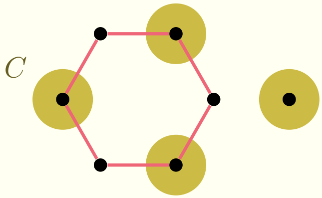
*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*



# How to find cospectral graphs

**Theorem (Godsil and McKay, 1982)**

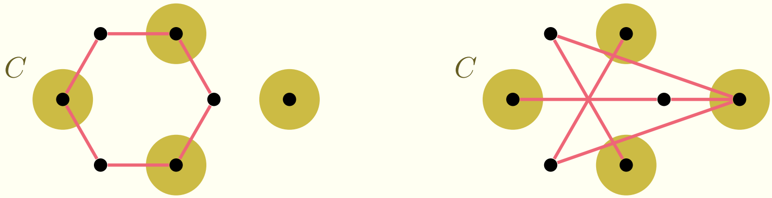
*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*



# How to find cospectral graphs

**Theorem (Godsil and McKay, 1982)**

*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*



# How to find cospectral graphs

**Theorem (Godsil and McKay, 1982)**

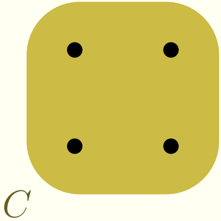
*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*

*Proof.*

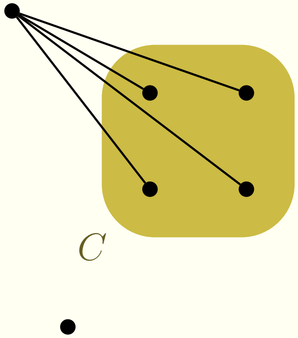
$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{2}J - I & O \\ O & I \end{pmatrix}.$$

□

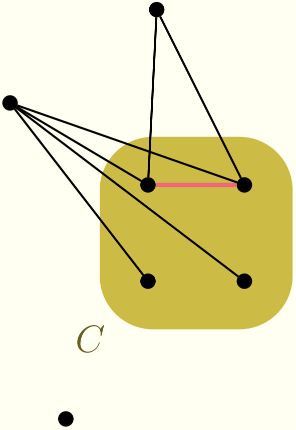
# How to find cospectral graphs



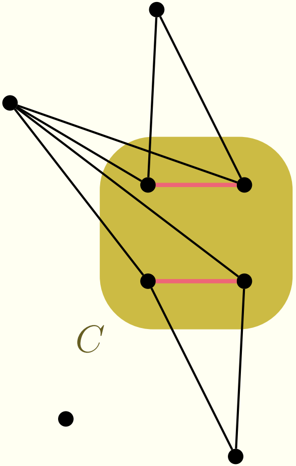
# How to find cospectral graphs



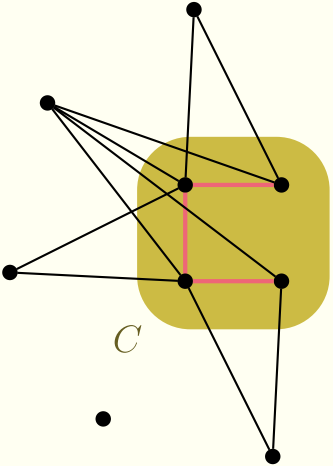
# How to find cospectral graphs



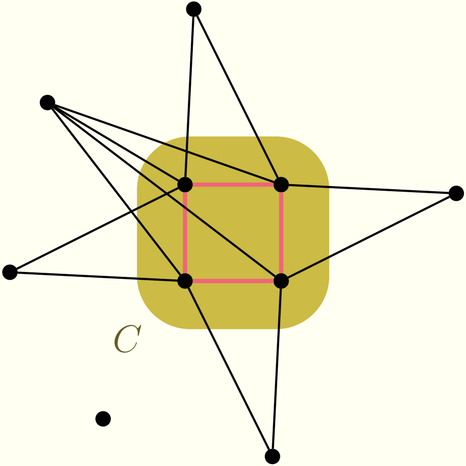
# How to find cospectral graphs



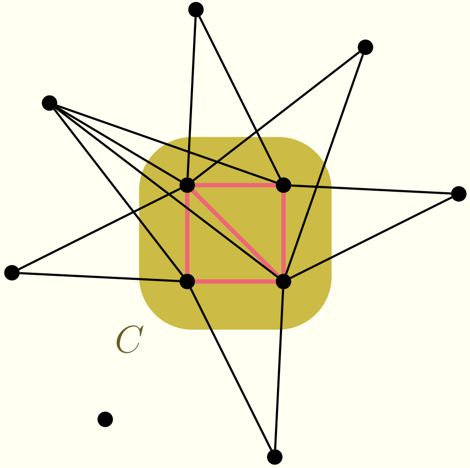
# How to find cospectral graphs



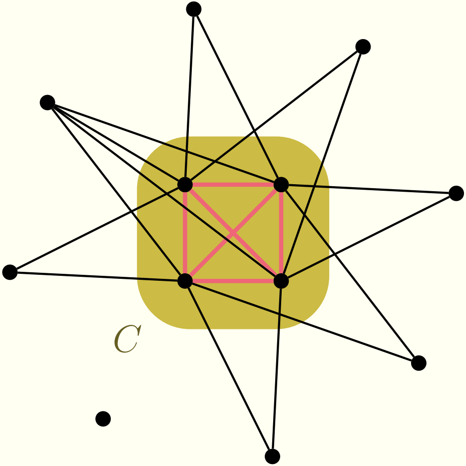
# How to find cospectral graphs



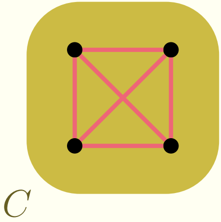
# How to find cospectral graphs



# How to find cospectral graphs



# How to find cospectral graphs




$\mathrm{AG}(2, 2)$

# How to find cospectral graphs

**Definition**

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a **switching set** with some conditions.

# How to find cospectral graphs

**Definition**

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a **switching set** with some conditions.

"We here define **switching** and **switches** as certain local transformations that do not alter the basic parameters of a combinatorial structure." [Östergård, *Switching codes and designs*, 2012]

# Table of contents

# Switching methods

**Definition**

A **switching method** is a graph operation, resulting in a cospectral graph. It needs a **switching set** with some conditions.

- GM-switching [Godsil and McKay, 1982]
- WQH-switching [Wang, Qiu and Hu, 2019]
- AH-switching [Abiad and Haemers, 2012]
  - Sun graph switching [Mao, Wang, Liu and Qiu, 2023]
  - Fano switching [Abiad, van de Berg and Simoens, 2025+]
  - Cube switching [Abiad, van de Berg and Simoens, 2025+]

# Level 2: Abiad-Haemers switching

**Theorem (Chan, Rodger and Seberry, 1986)**

*Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level* $2$ *and row sum* $1$ *is one of the following:*

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\ (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

$$(iii)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},\ (iv)\ \frac{1}{2}\begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

*where* $I$, $J$, $O$, $Y = 2I - J$ *and* $Z = J - I$, *are* $2 \times 2$ *matrices.*

# Level 2: Abiad-Haemers switching

**Theorem (Chan, Rodger and Seberry, 1986)**

*Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level $2$ and row sum $1$ is one of the following:*

GM-switching

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\quad (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

$$(iii)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},\quad (iv)\ \frac{1}{2}\begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

*where $I, J, O, Y = 2I - J$ and $Z = J - I$, are $2 \times 2$ matrices.*

# Level 2: Abiad-Haemers switching

**Theorem (Chan, Rodger and Seberry, 1986)**

*Up to permutations of rows and columns, an indecomposable regular orthogonal matrix of level* $2$ *and row sum* $1$ *is one of the following:*

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\quad (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

Fano switching

$$(iii)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},\quad (iv)\ \frac{1}{2}\begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

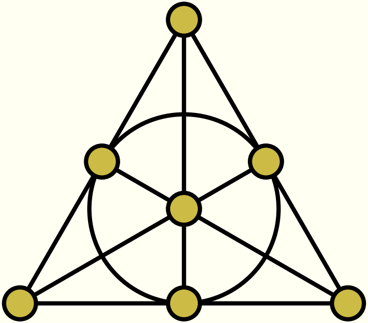
*where* $I$, $J$, $O$, $Y = 2I - J$ *and* $Z = J - I$, *are* $2 \times 2$ *matrices.*

# Fano switching

Abiad and Haemers (2012): algebraic conditions such that a conjugation of the adjacency matrix with $Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$, where

$$R = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

results in another adjacency matrix.



$PG(2, 2)$

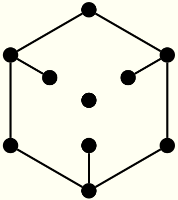
# Fano switching

**Theorem**

*Let $\Gamma$ be a graph with a subgraph $C$ whose vertices are identified as points of the Fano plane such that:*

- *$C$ is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in $C$.*
  - *If $x$ has 3 neighbours in $C$, they form a line.*
  - *If $x$ has 4 neighbours in $C$, they form the complement of a line.*

*Let $\pi$ be a permutation of the lines. For every $x \notin C$ that is (non)adjacent to the vertices of $\ell$, make it (non)adjacent to the vertices of $\pi(\ell)$. The resulting graph is cospectral with $\Gamma$.*



# Fano switching

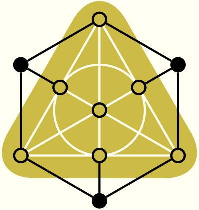
**Theorem**

*Let $\Gamma$ be a graph with a subgraph $C$ whose vertices are identified as points of the Fano plane such that:*

- *$C$ is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in $C$.*
  - *If $x$ has 3 neighbours in $C$, they form a line.*
  - *If $x$ has 4 neighbours in $C$, they form the complement of a line.*

*Let $\pi$ be a permutation of the lines. For every $x \notin C$ that is (non)adjacent to the vertices of $\ell$, make it (non)adjacent to the vertices of $\pi(\ell)$. The resulting graph is cospectral with $\Gamma$.*

$C$



# Fano switching

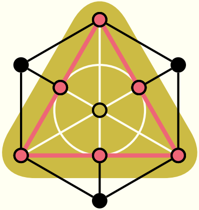
## Theorem

*Let $\Gamma$ be a graph with a subgraph $C$ whose vertices are identified as points of the Fano plane such that:*

- *$C$ is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in $C$.*
  - *If $x$ has 3 neighbours in $C$, they form a line.*
  - *If $x$ has 4 neighbours in $C$, they form the complement of a line.*

*Let $\pi$ be a permutation of the lines. For every $x \notin C$ that is (non)adjacent to the vertices of $\ell$, make it (non)adjacent to the vertices of $\pi(\ell)$. The resulting graph is cospectral with $\Gamma$.*

$C$



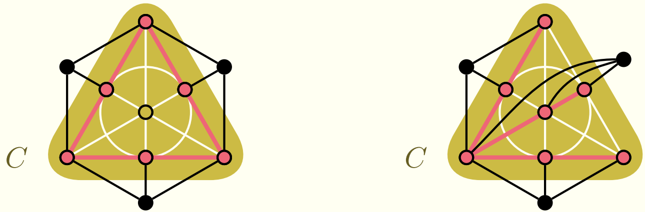
# Fano switching

**Theorem**

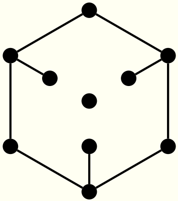
*Let $\Gamma$ be a graph with a subgraph $C$ whose vertices are identified as points of the Fano plane such that:*

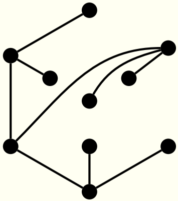
- *$C$ is edgeless or complete.*
- *Every vertex $x \notin C$ has 0, 3, 4 or 7 neighbours in $C$.*
  - *If $x$ has 3 neighbours in $C$, they form a line.*
  - *If $x$ has 4 neighbours in $C$, they form the complement of a line.*

*Let $\pi$ be a permutation of the lines. For every $x \notin C$ that is (non)adjacent to the vertices of $\ell$, make it (non)adjacent to the vertices of $\pi(\ell)$. The resulting graph is cospectral with $\Gamma$.*

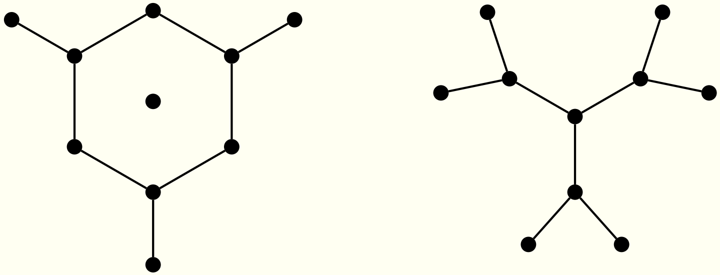


# Fano switching





# Fano switching



Both graphs have spectrum $\{(-\sqrt{5})^1, (-\sqrt{2})^2, (0)^3, (\sqrt{2})^2, (\sqrt{5})^1\}$.

# Design switching

**Definition**

An $(r, \lambda)$-**design** is a design where every point is contained in $r$ blocks and every two points are contained in $\lambda$ blocks.

# Design switching

**Definition**

An $(r, \lambda)$-**design** is a design where every point is contained in $r$ blocks and every two points are contained in $\lambda$ blocks.

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with an edgeless or complete subgraph $C$ whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)|\,.$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

# Design switching

**Definition**

An $(r, \lambda)$-**design** is a design where every point is contained in $r$ blocks and every two points are contained in $\lambda$ blocks.

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with an edgeless or complete subgraph $C$ whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)|\,.$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

# Design switching

## Definition

An $(r, \lambda)$-**design** is a design where every point is contained in $r$ blocks and every two points are contained in $\lambda$ blocks.

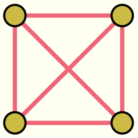
## Theorem (Ihringer and Simoens, 2025+)

*Let $\Gamma$ be a graph with an edgeless or complete subgraph $C$ whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| \,.$$

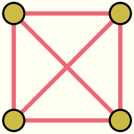
*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

# Design switching



is an $(r = 3, \lambda = 1)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

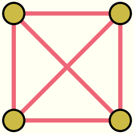


is an $(r = 3, \lambda = 1)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

# Design switching



is an $(r = 3, \lambda = 1)$-design with incidence matrix

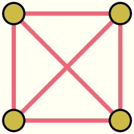
| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ |
|---|---|---|---|---|---|---|
| $p_1$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $p_2$ | 1 | 0 | 0 | 1 | 1 | 0 |
| $p_3$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $p_4$ | 0 | 0 | 1 | 0 | 1 | 1 |

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

**Theorem (Godsil and McKay, 1982)**

*Let $\Gamma$ be a graph with a regular subgraph of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*

# Design switching



is an $(r = 3, \lambda = 1)$-design with incidence matrix

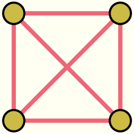
$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

**Theorem (Godsil and McKay, 1982)**

*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*

# Design switching



is an $(r = 3, \lambda = 1)$-design with incidence matrix

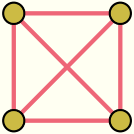
$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

**Theorem (Godsil and McKay, 1982)**

*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*

# Design switching



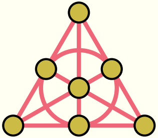
is an $(r = 4, \lambda = 2)$-design with incidence matrix

$$
\begin{array}{c}
\\
p_1 \\ p_2 \\ p_3 \\ p_4
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 &
\end{array}\\
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}.
$$

$\pi : B_i \mapsto B_{7-i}$ preserves pairwise intersection.

**Theorem (Godsil and McKay, 1982)**

*Let $\Gamma$ be a graph with a regular subgraph $C$ of size $4$ such that every vertex $x \notin C$ has $0, 2$ or $4$ neighbours in $C$. For every $x \notin C$ that has exactly $2$ neighbours in $C$, reverse its adjacencies with $C$. The resulting graph is cospectral with $\Gamma$.*

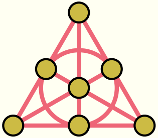


is an $(r = 3, \lambda = 1)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array}\begin{array}{c} \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Any permutation of the lines $\pi$ preserves pairwise intersection.

# Design switching

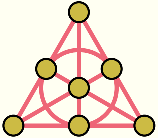


is an $(r = 8, \lambda = 4)$-design with incidence matrix

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Any permutation of the lines $\pi$ preserves pairwise intersection.

➤ Fano switching



is an $(r = 8, \lambda = 4)$-design with incidence matrix

| | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $\overline{B_1}$ | $\overline{B_2}$ | $\overline{B_3}$ | $\overline{B_4}$ | $\overline{B_5}$ | $\overline{B_6}$ | $\overline{B_7}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $p_2$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| $p_3$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| $p_4$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| $p_5$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| $p_6$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| $p_7$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |

Any permutation of the lines $\pi$ preserves pairwise intersection.

- Fano switching

# Design switching

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with an edgeless or complete subgraph $C$ whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)|\,.$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

# Design switching

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with an edgeless or complete subgraph $C$ whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| .$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

*Proof.* Define $R = \frac{1}{r-\lambda}\left(N(N^\pi)^T - \lambda J\right)$, where $N^\pi$ is obtained from the incidence matrix $N$ by permuting the columns with $\pi$.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} R & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R & O \\ O & I \end{pmatrix} .$$

□

# Design switching

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with a subgraph $C$ with adjacency matrix $A_C$ such that $R^T A_C R$ is again an adjacency matrix whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| \, .$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

*Proof.* Define $R = \frac{1}{r-\lambda}\left(N(N^\pi)^T - \lambda J\right)$, where $N^\pi$ is obtained from the incidence matrix $N$ by permuting the columns with $\pi$.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} R & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R & O \\ O & I \end{pmatrix} .$$

□

# Design switching

**Theorem (Ihringer and Simoens, 2025+)**

*Let $\Gamma$ be a graph with a subgraph $C$ with adjacency matrix $A_C$ such that $R^T A_C R$ is again an adjacency matrix whose vertices are identified as points of an $(r, \lambda)$-design such that every vertex $x \notin C$ is adjacent to the points of a block or its complement.*
*Let $\pi$ be a permutation of the blocks such that for all blocks $B_i, B_j$,*

$$|B_i \cap B_j| = |\pi(B_i) \cap \pi(B_j)| .$$

*For every $x \notin C$ adjacent to the points of $B$, make it adjacent to the points of $\pi(B)$. The resulting graph is cospectral with $\Gamma$.*

*Proof.* Define $R = \frac{1}{r-\lambda}\left(N(N^\pi)^T - \lambda J\right)$, where $N^\pi$ is obtained from the incidence matrix $N$ by permuting the columns with $\pi$.

$$\begin{pmatrix} A_{11} & A'_{12} \\ A'_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} R & O \\ O & I \end{pmatrix}^T \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} R & O \\ O & I \end{pmatrix} .$$

□

# Table of contents

# Triangular graphs

## Definition

The **triangular graph** $T_n$ has as vertices the $2$-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if they intersect.

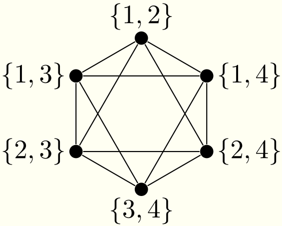
In other words, $T_n = L(K_n)$.

# Triangular graphs

**Definition**

The **triangular graph** $T_n$ has as vertices the $2$-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if they intersect.

In other words, $T_n = L(K_n)$.



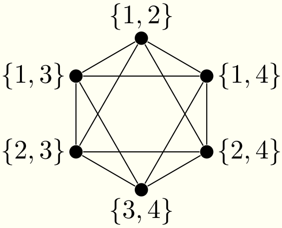

The octahedral graph $T_4$.

# Triangular graphs

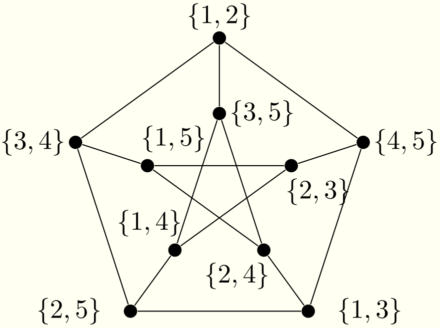
**Definition**

The **triangular graph** $T_n$ has as vertices the 2-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if they intersect.

In other words, $T_n = L(K_n)$.




The octahedral graph $T_4$.




The Petersen graph $\overline{T_5}$.

# Triangular graphs

**Theorem (Chang and Hoffman, independently, 1959)**

*The triangular graph $T_n$ is determined by its spectrum iff $n \neq 8$.*

# q-triangular graphs

**Definition**

The **q-triangular graph** $T_{q,n}$ has as vertices the 2-dimensional subspaces of $\mathbb{F}_q^n$ where two vertices are adjacent if they intersect.

# q-triangular graphs

**Definition**

The **q-triangular graph** $T_{q,n}$ has as vertices the lines of $\mathrm{PG}(n-1, q)$ where two vertices are adjacent if they intersect.

# q-triangular graphs

**Definition**

The **q-triangular graph** $T_{q,n}$ has as vertices the lines of $\mathrm{PG}(n–1, q)$ where two vertices are adjacent if they intersect.

**Theorem (Ihringer and Munemasa, 2019)**

*The $q$-triangular graph $T_{q,n}$ is not determined by its spectrum if $n \geq 4$.*

# q-triangular graphs

**Definition**

The **q-triangular graph** $T_{q,n}$ has as vertices the lines of $\mathrm{PG}(n–1, q)$ where two vertices are adjacent if they intersect.

**Theorem (Ihringer and Munemasa, 2019)**

*The $q$-triangular graph $T_{q,n}$ is not determined by its spectrum if $n \geq 4$.*

*Proof.* Consider the subgraph $T_{q,3}$ of all lines in a given plane $\mathrm{PG}(2, q) \subseteq \mathrm{PG}(n-1, q)$ and consider the design $D = (\mathcal{P}, \mathcal{B})$ where

$$\mathcal{P} = \{\text{lines of } \mathrm{PG}(2, q)\}$$
$$\mathcal{B} = \{\text{point pencils of } \mathrm{PG}(2, q)\}$$

Apply design switching, using any permutation $\pi$ of $\mathcal{B}$ that is not an automorphism. This creates maximal cliques of size $q^2 + q$. □

# q-triangular graphs

**Theorem (Ihringer and Simoens, 2025+)**

*There are at least $q!$ graphs with the same spectrum as $T_{q,n}$.*

# q-triangular graphs

**Theorem (Ihringer and Simoens, 2025+)**

*There are at least $q!$ graphs with the same spectrum as $T_{q,n}$.*

*Proof (same strategy as in [Brouwer, Ihringer and Kantor, 2022]).*
Let $\Gamma_\pi$ denote the graph obtained from design switching $T_{n,q}$ with $\pi$. Then

$$\Gamma_{\pi_1} \cong \Gamma_{\pi_2}$$
$$\iff \pi_1 \text{ and } \pi_2 \text{ are in the same double coset of } \mathrm{Aut}(D) \text{ in } \mathrm{Sym}(\mathcal{B}).$$

There are at least $q!$ double cosets. □

# q-triangular graphs

**Theorem (Ihringer and Simoens, 2025+)**

*There are at least $q!$ graphs with the same spectrum as $T_{q,n}$.*

*Proof (same strategy as in [Brouwer, Ihringer and Kantor, 2022]).*
Let $\Gamma_\pi$ denote the graph obtained from design switching $T_{n,q}$ with $\pi$. Then

$$\Gamma_{\pi_1} \cong \Gamma_{\pi_2}$$
$$\iff \pi_1 \text{ and } \pi_2 \text{ are in the same double coset of } \mathrm{Aut}(D) \text{ in } \mathrm{Sym}(\mathcal{B}).$$

There are at least $q!$ double cosets. □

- ➤ Many strongly regular graphs with the same parameters.

# Ongoing work

- Many designs to try

# Ongoing work

- Many designs to try
- Alternative proofs of cospectrality results
  - q-triangular graphs [Ihringer and Munemasa, 2019]
  - Collinearity graphs of polar spaces [Brouwer, Ihringer and Kantor, 2022]
  - Collinearity graphs of generalised quadrangles [Guo and van Dam, 2022]

# Ongoing work

- Many designs to try
- Alternative proofs of cospectrality results
  - q-triangular graphs [Ihringer and Munemasa, 2019]
  - Collinearity graphs of polar spaces [Brouwer, Ihringer and Kantor, 2022]
  - Collinearity graphs of generalised quadrangles [Guo and van Dam, 2022]
- All commonly known *indecomposable* switching methods can be reformulated as design switching.
- More general: $\pi$ may also be a bijection between blocks of different designs.

Thank you for listening!