Combinatorics of finite spherical buildings

Sam Mattheus

5th Pythagorean Conference
June 3, 2025

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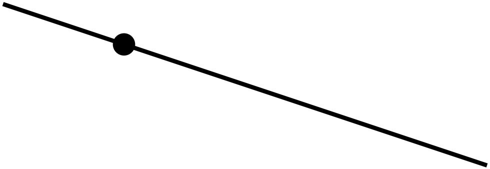
Jacques Tits (1930-2021)

Foundations



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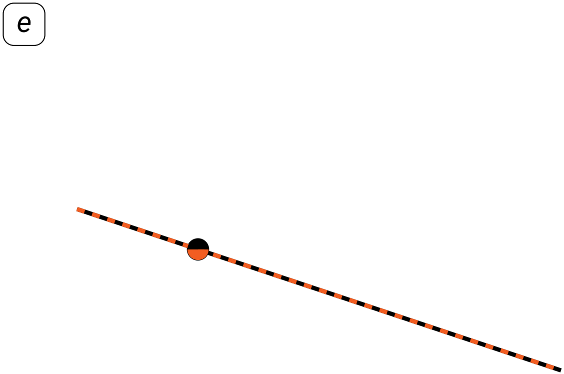
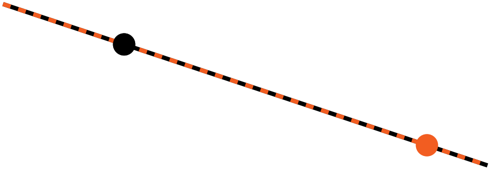


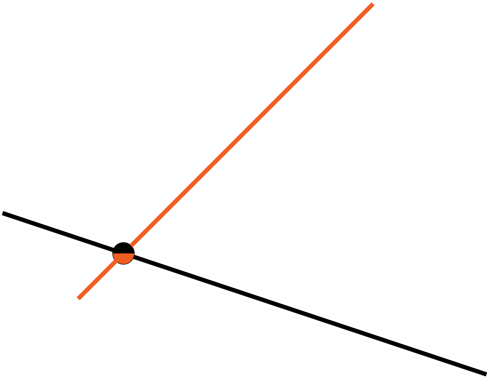
Foundations

1



Foundations

2



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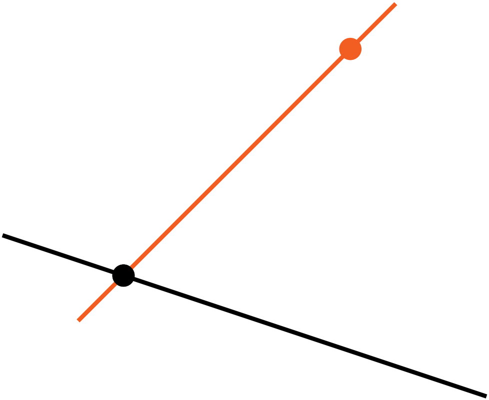
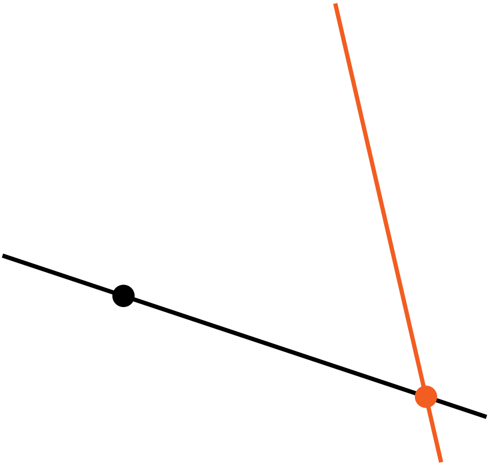
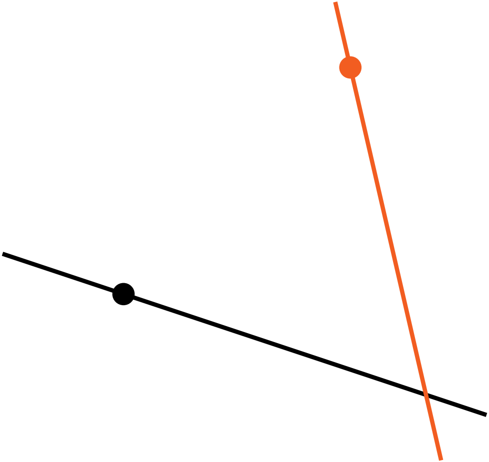


12

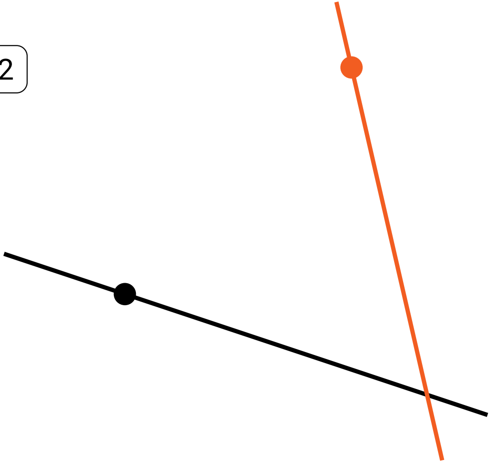


121



Foundations

$121 = 212$



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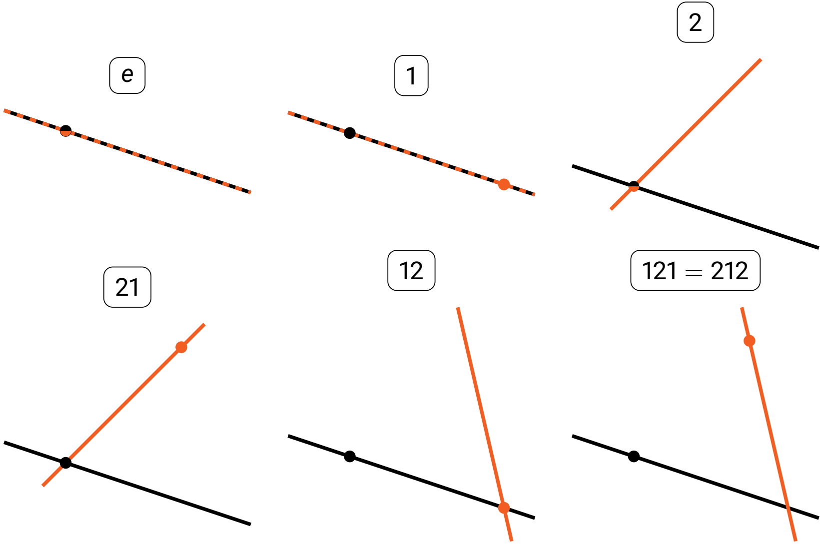

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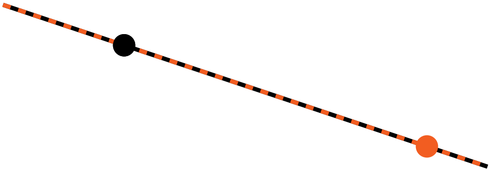

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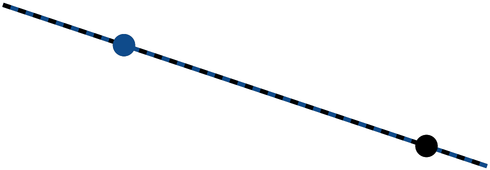
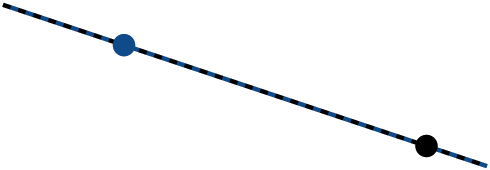


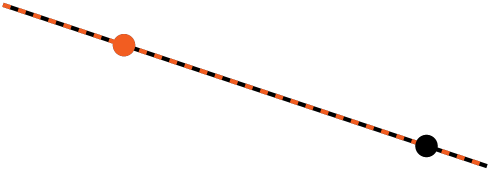
Foundations

$1 * 1$



Foundations

$$1 * 1 = e$$



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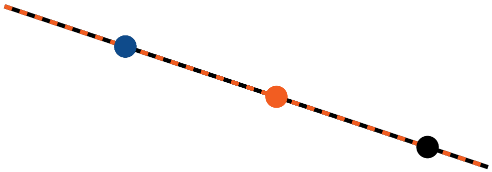
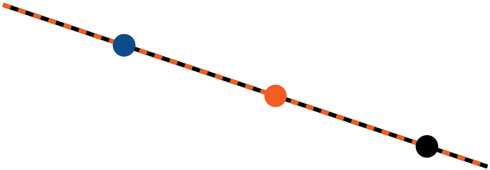
$$1 * 1 \in \{e, 1\}$$



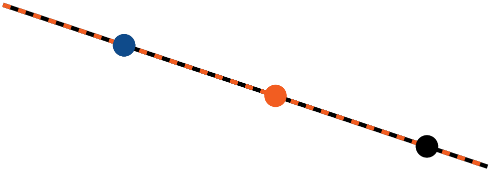
Foundations

$$1 * 1 = 1 \cdot e + (q - 1) \cdot 1$$



Foundations

$$A_1 * A_1 = 1 \cdot A_e + (q-1) \cdot A_1$$



Foundations

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$A_1 * A_2 = A_{12}$

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$$A_{12} * A_1 = A_{121}$$

$$A_{21} * A_1 = 1 \cdot A_2 + (q-1) \cdot A_{21}$$

Foundations

Definition

On the set of **words** (finite sequences of 1s and 2s) we define

- $11 = 22 = e$ (involutions)
- $121 = 212$ (the braid rule)

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A word is **reduced** if any combination of the rules do not shorten it.

The **length** $\ell(w)$ of a reduced word is what you think it is.

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The reduced words and their lengths are

e	1	2	12	21	121
0	1	1	2	2	3

Foundations

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Let W denote the set of reduced words.
(W, concatenation + reduction) is a group!

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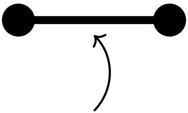

$$121212 = (12)^3 = e$$

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(W, concatenation + reduction) is a group!

$$W = \langle 1, 2 \mid 11 = 22 = e, 121 = 212 \rangle.$$

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$$(W, \text{concatenation + reduction}) \cong \mathrm{Sym}(3)$$

$$1 \leftrightarrow (12)$$
$$2 \leftrightarrow (23)$$

Foundations

Observation

For any $w \in W$ and $i \in \{1, 2\}$

$$A_w * A_i = \begin{cases} A_{wi} & \text{if } \ell(wi) > \ell(w) \\ A_{wi} + (q-1)A_w & \text{if } \ell(wi) < \ell(w). \end{cases}$$

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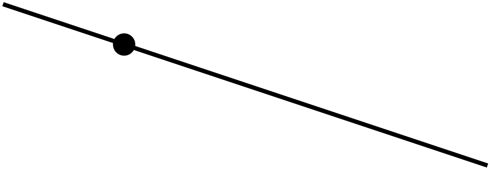
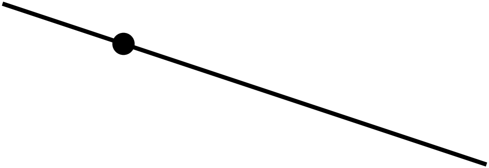
For $q = 1$ we retrieve $\mathrm{Sym}(3)$!

Foundations



Foundations

$$(\langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle)$$



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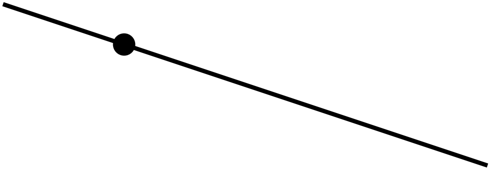
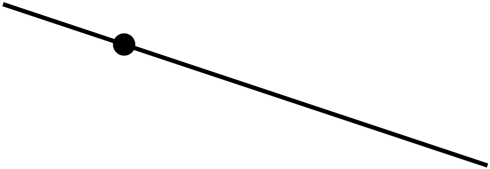
$$((\langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle) \curvearrowleft (12) \leftrightarrow 1$$



Foundations

$$(\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle) \curvearrowleft (12) \leftrightarrow 1$$
$$= (\langle e_2\rangle, \langle e_2, e_1\rangle, \langle e_2, e_1, e_3\rangle)$$



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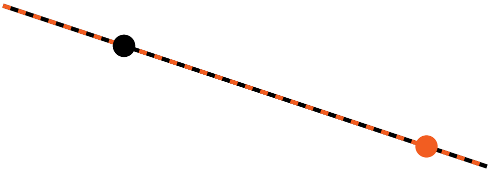

Foundations

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Given a fixed flag

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The set $\{A_w\}_{w \in W}$ generates a (non-commutative) matrix algebra over $\mathbb{C}$ satisfying

1. $A_e = I$,
2. $\sum_{w \in W} A_w = J$,
3. $(A_w)^T = A_{w^{-1}}$,

Foundations

Summary

We have

- a geometry of flags

Foundations

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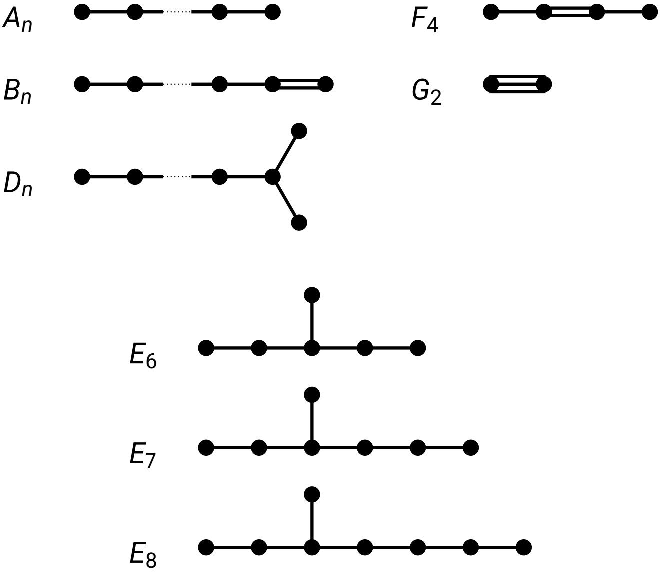
- a Dynkin diagram
- an Iwahori-Hecke algebra
- a classical group

Spherical buildings



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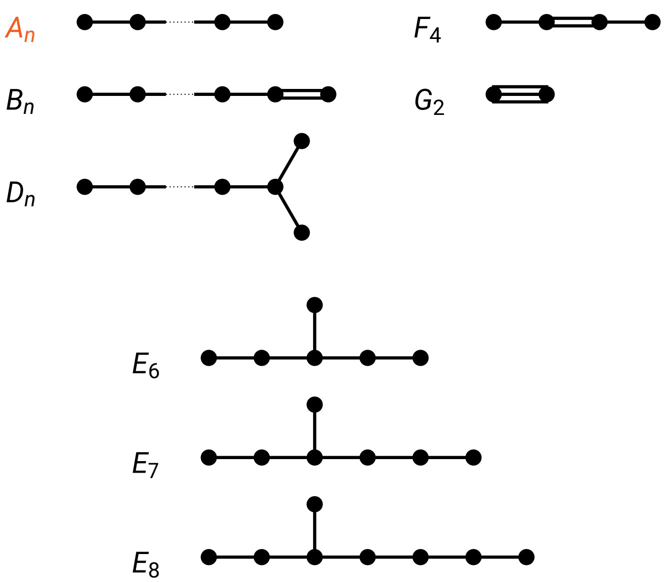

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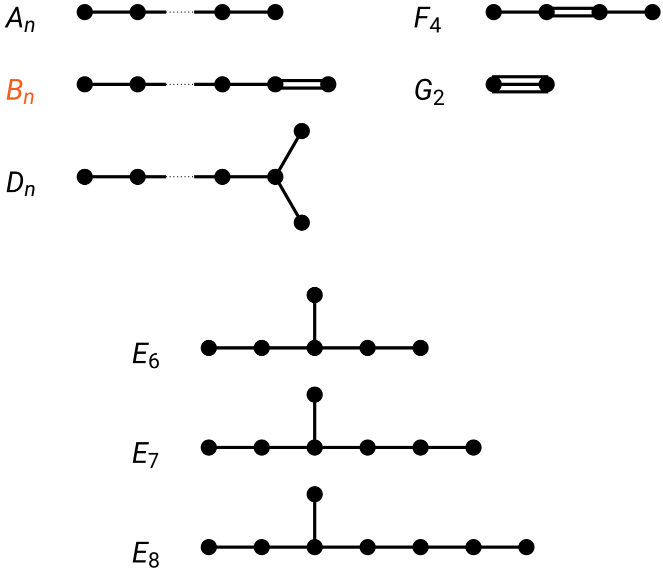

Oppositeness

Fact

There is a unique word w_0 of longest length.

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Two flags are **opposite** if they are in relation w_0.

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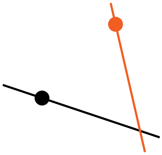
Example

In A_2, $w_0 = 121$



Oppositeness

Problem

What is the largest set of pairwise non-opposite flags?

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Theorem (Erdős-Ko-Rado 1961)

The largest family of pairwise intersecting k-subsets of an n-set, $n \geq 2k$, has size at most

$$\binom{n-1}{k-1}.$$

For $n > 2k$, equality is attained only by stars.

Algebraic proofs

Ratio bound

Let S be an independent set in a d-regular graph on v vertices whose smallest eigenvalue is λ. Then

$$|S| \leq \frac{-v\lambda}{d-\lambda}.$$

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Kneser graph

Vertex set = k-subsets of an n-set,
Edge set = $A \sim B$ if $A \cap B = \emptyset$.

$$v = \binom{n}{k}, \qquad d = \binom{n-k}{k}, \qquad \lambda = -\binom{n-k-1}{k-1}.$$

Algebraic proofs

Ratio bound

Let S be an independent set in a d-regular graph on v vertices whose smallest eigenvalue is λ. Then

$$|S| \leq \frac{-v\lambda}{d-\lambda}.$$

Moreover if equality is attained then $1_S \in E_d \oplus E_\lambda$.

Algebraic proof?

Bad news

The Iwahori-Hecke algebra is not commutative.

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The Iwahori-Hecke algebra is not commutative.

Good news

$A_{w_0}^2$ is central in it.

Algebraic proof!

Theorem (De Beule-M.-Metsch 2022)

Let S be a set of pairwise non-opposite flags and $n \geq 2$, then

$$|S| \leq \frac{\#\text{ of flags}}{q^{(n+1)/2}+1} \qquad \text{in type } A_n,$$

$$|S| \leq \frac{\#\text{ of flags}}{q^{n+e-1}+1} \qquad \text{in (most of) type } B_n.$$

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Theorem (De Beule-M.-Metsch 2025+)

We have a description of a spanning set for E_λ in

- *type A_{2n+1}, $n \geq 1$*
- *(most of) type B_n, $n \geq 2$.*

Algebraic proof!

Theorem (Heering-Lansdown-Metsch 2025)

For q large enough, equality in type A_{2n+1} is attained only by blow-ups of (dual) stars of n-spaces.

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Theorem (De Beule-Heering-M.-Metsch 2025+)

For q large enough, equality (in most cases) in type B_n is attained only by blow-ups of the set of points in a generator, or a blow-up of a star of generators, Thank you for your attention!

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