Stable Higher Specht Polynomials

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Notation

S_n - The symmetric group on n elements.
$S_{\mathbb{N}}$ - The symmetric group acting on $\mathbb{N}$
$S_\infty := \{\sigma \in S_{\mathbb{N}} \mid \exists n\ \sigma(m) = m\ \forall m > n\}$
$\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$
$\mathbb{Q}[\mathbf{x}_\infty] := \mathbb{Q}[x_1, x_2, \ldots]$
$\mathbb{Q}[\![\mathbf{x}_\infty]\!] := \mathbb{Q}[\![x_1, x_2, \ldots]\!]$ (formal power series)
Λ - The ring of symmetric functions. We thus have:

$$\Lambda = \{F \in \mathbb{Q}[\![\mathbf{x}_\infty]\!] \mid \sigma F = F\ \forall \sigma \in S_{\mathbb{N}},\ \deg F < \infty\}.$$

Actions and Representations

The group S_n acts on $\mathbb{Q}[\mathbf{x}_n]$ by interchanging the variables.
Similarly, $S_{\mathbb{N}}$ acts on $\mathbb{Q}[\mathbf{x}_\infty]$, as does its subgroup S_∞. They also act trivially on elements of Λ.
Both $\mathbb{Q}[\mathbf{x}_n]$ and $\mathbb{Q}[\mathbf{x}_\infty]$ are graded rings, with the usual grading. The actions preserve degrees.
$\mathbb{Q}[\mathbf{x}_n]_d$ - The part of $\mathbb{Q}[\mathbf{x}_n]$ that is homogeneous of degree d.
Similarly $\mathbb{Q}[\mathbf{x}_\infty]_d$ and Λ_d.

A Phenomenon

$$\mathbb{Q}[\mathbf{x}_n]_1^0 := \left\{ \sum_{i=1}^{n} a_i x_i \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\} \subseteq \mathbb{Q}[\mathbf{x}_n]_1.$$

It is an irreducible sub-representation, and we have $\mathbb{Q}[\mathbf{x}_n]_1 = \mathbb{Q}[\mathbf{x}_n]_1^0 \oplus \mathbb{Q}e_1^{(n)}$, where $e_1^{(n)} := \sum_{i=1}^{n} x_i$. Same $\forall n \geqslant 2$.

$$\mathbb{Q}[\mathbf{x}_\infty]_1^0 := \left\{ \sum_{i=1}^{\infty} a_i x_i \in \mathbb{Q}[\mathbf{x}_\infty]_1 \;\middle|\; \sum_{i=1}^{\infty} a_i = 0 \right\}.$$

It is irreducible, of co-dimension 1, but with no complement. The element e_1 is no longer in $\mathbb{Q}[\mathbf{x}_\infty]$, but rather in Λ.

Partitions and Tableaux

$\lambda \vdash n$ indicates that λ is a partition of n (or its Ferrers diagram).
$\mathrm{SYT}(\lambda)$ - The set of standard Young tableaux of shape λ (and content $\{1, \dots, n\}$).
$\mathrm{SSYT}(\lambda)$ - The set of standard Young tableaux of shape λ, containing non-negative integers.
For example, $\lambda = 431 \vdash 8$, then $T \in \mathrm{SYT}(\lambda)$ and $M \in \mathrm{SSYT}(\lambda)$ with

$$T := \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}, \qquad M := \begin{array}{|c|c|c|c|} \hline 0 & 0 & 2 & 5 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}.$$

The Irreducible Representations of S_n

Theorem (classical)

The irreducible representations of S_n are in one-to-one correspondence with partitions of n. The representation $\mathcal{S}^\lambda$ that is associated with $\lambda \vdash n$ has dimension $f^\lambda := |\mathrm{SYT}(\lambda)|$.

The abstract representation $\mathcal{S}^\lambda$ (or its classical construction using classes of tableaux) is called a *Specht module*.
Higher Specht polynomials, and their generalizations, span realizations of Specht modules as representations inside $\mathbb{Q}[\mathbf{x}_n]$.

Row and Column Subgroups

Fix $\lambda \vdash n$ and $T \in \mathrm{SYT}(\lambda)$. Then there are subgroups $R(T) := \{\tau \in S_n \mid \tau \text{ preserves the rows of } T\}$ and $C(T) := \{\sigma \in S_n \mid \sigma \text{ preserves the columns of } T\}$.
The module $\mathcal{S}^\lambda$ is spanned by $\{\mathbf{e}_T \mid T \in \mathrm{SYT}(\lambda)\}$, where e_T is the *polytabloid* associated with T. We have $\sigma \mathbf{e}_T = \mathrm{sgn}(\sigma)\mathbf{e}_T$ for $\sigma \in C(T)$.

$T :=$

1	2	4	7
3	5	8	
6			

$R(T) = S_{\{1,2,4,7\}} \times S_{\{3,5,8\}} \subseteq S_8$
$C(T) = S_{\{1,3,6\}} \times S_{\{2,5\}} \times S_{\{4,8\}} \subseteq S_8$

Specht Polynomials

$$T := \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \cline{1-3} 6 & & & \\ \cline{1-1}\end{array}$$

$R(T) = S_{\{1,2,4,7\}} \times S_{\{3,5,8\}} \subseteq S_8$
$C(T) = S_{\{1,3,6\}} \times S_{\{2,5\}} \times S_{\{4,8\}} \subseteq S_8$
$\sigma \mathbf{e}_T = \mathrm{sgn}(\sigma)\mathbf{e}_T$ for $\sigma \in C(T)$.
A natural polynomial is then the *Specht polynomial*

$$F_{C^0,T} := \prod_{c \in \mathrm{Col}(T)} \prod_{i<j \in c} (x_j - x_i)$$

In our example it is $(x_3 - x_1)(x_6 - x_1)(x_6 - x_3)(x_5 - x_2)(x_8 - x_4)$.

Representations on Specht Polynomials

Theorem (old - Specht, Peel, Murphy)

Given $\lambda \vdash n$, set $F_{C^0,T} := \prod_{c \in \mathrm{Col}(T)} \prod_{i<j \in c} (x_j - x_i)$ for every $T \in \mathrm{SYT}(\lambda)$. Then $\{\mathbf{F}_{C^0,T} \mid T \in \mathrm{SYT}(\lambda)\}$ span a representation of S_n that is isomorphic to $\mathcal{S}^\lambda$.

The higher Specht polynomials produce additional copies of $\mathcal{S}^\lambda$ inside $\mathbb{Q}[\mathbf{x}_n]$, originally by Ariki, Terasoma, and Yamada to obtain an explicit expression for the co-invariant ring as a regular representation of S_n. We define directly the generalized higher Specht polynomials.

For Generalized Higher Specht Polynomials

Take $\lambda \vdash n$, $T \in \mathrm{SYT}(\lambda)$, and $M \in \mathrm{SSYT}(\lambda)$.
$v_T(i)$ - The box containing an index $1 \leqslant i \leqslant n$ in T.
$M[v]$ - The value showing up in M inside the box $v \in \lambda$.
The *Young symmetrizer* is $\varepsilon_T := \sum_{\sigma \in C(T)} \sum_{\tau \in R(T)} \mathrm{sgn}(\sigma)\sigma\tau$.
We define $p_{M,T} := \prod_{i=1}^{n} x_i^{M[v_T(i)]}$. In our example

$$T := \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}, \qquad M := \begin{array}{|c|c|c|c|}\hline 0 & 0 & 2 & 5 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array},$$

we have $p_{M,T} = x_3 x_4^2 x_5^2 x_6^4 x_7^5 x_8^4$.

Generalized Higher Specht Polynomials

Definition (Generalized Higher Specht Polynomials)

Set $p_{M,T} := \prod_{i=1}^{n} x_i^{M[v_T(i)]}$, write $s_{M,T}$ for the size of the stabilizer of $p_{M,T}$ in $R(T)$, and define $F_{M,T} := \varepsilon_T p_{M,T}/s_{M,T}$.

When C^0 is the minimal element of $\mathrm{SSYT}(\lambda)$, this reproduces $F_{C^0,T}$.

Theorem (Peel, Murphy, Gillespie–Rhoades)

The space V_M spanned by $\{F_{M,T} \mid T \in \mathrm{SYT}(\lambda)\}$ is a representation of S_n that is isomorphic to $\mathcal{S}^\lambda$ by $F_{M,T} \leftrightarrow \mathbf{e}_T$.

Dividing by $s_{M,T}$ makes $F_{M,T}$ primitive in $\mathbb{Z}[\mathbf{x}_n]$. This scalar depends on M but not on T, and just rescales the representation at this point.

Extended Column Subgroups

Recall that $C(T) := \{\sigma \in S_n \mid \sigma \text{ preserves the columns of } T\}$.
Write $\tilde{C}(T)$ for the group containing $C(T)$ but in which full columns of the same size can also be permuted. It is the normalizer $N_{S_n}\big(C(T)\big)$, and a semi-direct product $\big(R(T) \cap \tilde{C}(T)\big) \rtimes C(T)$.
The sign character on $C(T)$ and the trivial character on $R(T) \cap \tilde{C}(T)$ combine to a character $\widetilde{\mathrm{sgn}} : \tilde{C}(T) \to \{\pm 1\}$.

$T :=$

1	2	4	7
3	5	8	
6			

$C(T) = S_{\{1,3,6\}} \times S_{\{2,5\}} \times S_{\{4,8\}} \subseteq S_8$
$R(T) \cap \tilde{C}(T) = \langle (24)(58) \rangle$, $\tilde{C}(T)$ is the semi-direct product.

Actions of Extended Column Subgroups

Let $\lambda \vdash n$, $T \in \mathrm{SYT}(\lambda)$, and $M \in \mathrm{SSYT}(\lambda)$ be given. Recall the group $\tilde{C}(T) = \big(R(T) \cap \tilde{C}(T)\big) \rtimes C(T)$ and its character $\widetilde{\mathrm{sgn}} : \tilde{C}(T) \to \{\pm 1\}$.

Proposition

The generalized higher Specht polynomial $F_{M,T}$ is in the $\widetilde{\mathrm{sgn}}$-eigenspace of the action of $\tilde{C}(T)$.

Corollary

The quotient $Q_{M,T} := F_{M,T}/F_{C^0,T}$ is $\tilde{C}(T)$-invariant.

Conjecture

The symmetric functions inside $\mathbb{Q}[\mathbf{x}_n]$ and the quotients $Q_{C,T}$, where C runs over the cocharge tableaux of shape λ, generate the ring of $\tilde{C}(T)$-invariants inside $\mathbb{Q}[\mathbf{x}_n]$.

Extending Tableaux

Definition

- For $\lambda \vdash n$, let $\lambda_+ \vdash n+1$ be λ plus another box in the first row.
- Given $T \in \mathrm{SYT}(\lambda)$, we denote by $\iota T \in \mathrm{SYT}(\lambda_+)$ the tableau filled as T plus $n+1$ in the new box.
- If $M \in \mathrm{SSYT}(\lambda)$ then $\hat{\iota} M \in \mathrm{SSYT}(\lambda_+)$ is obtained from M by shoving the first row one box to the right, and filling with a 0.

$$T := \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}, \qquad M := \begin{array}{|c|c|c|c|}\hline 0 & 0 & 2 & 5 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}.$$

$$\iota T := \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 9 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array}, \qquad \hat{\iota} M := \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 2 & 5 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}.$$

The Key Relation

Let $\lambda \vdash n$, $T \in \mathrm{SYT}(\lambda)$, and $M \in \mathrm{SSYT}(\lambda)$ be given, and set $\lambda_+ \vdash n+1$, $\iota T \in \mathrm{SYT}(\lambda_+)$, and $\hat{\iota} M \in \mathrm{SSYT}(\lambda_+)$ be as above.
We then have $F_{M,T} \in \mathbb{Q}[\mathbf{x}_n]$ and $F_{\hat{\iota}M,\iota T} \in \mathbb{Q}[\mathbf{x}_{n+1}]$.

Proposition

We have the equality $F_{\hat{\iota}M,\iota T}(x_{n+1} = 0) = F_{M,T}$. Moreover, if M contains enough zeros, then $F_{\hat{\iota}M,\iota T}$ is the only element of the $\widetilde{\mathrm{sgn}}$-eigenspace of $\tilde{C}(\iota T)$ which satisfies this equality.

The division by $s_{M,T}$ in the definition is crucial here.
By repeated applications, we get the following result.

Theorem

There is a homogeneous power series $F_{\hat{M},\hat{T}} \in \mathbb{Q}[\![\mathbf{x}_\infty]\!]$ that is symmetric in $\{x_m \mid m > n\}$ and satisfies $F_{\hat{M},\hat{T}}(x_m = 0\ \forall m > n) = F_{M,T}$.

The series $F_{\hat{M},\hat{T}}$ is an inverse limit of $\{F_{\hat{\iota}M,\iota T} \to F_{M,T}\}$.

Eventually Symmetric Functions

We denote by $\tilde{\Lambda}$ the ring of *eventually symmetric functions*. Namely if $S_{\mathbb{N}}^{(n)} := \{\sigma \in S_{\mathbb{N}} \mid \sigma(i) = i \ \forall 1 \leqslant i \leqslant n\}$ then

$$\tilde{\Lambda} := \{F \in \mathbb{Q}[\![\mathbf{x}_\infty]\!] \mid \exists n \text{ s.t. } \sigma F = F \ \forall \sigma \in S_{\mathbb{N}}^{(n)}, \ \deg F < \infty\}.$$

These are also known as *almost symmetric functions*.

Proposition

- The ring $\tilde{\Lambda}$ is generated inside $\mathbb{Q}[\![\mathbf{x}_\infty]\!]$ by $\mathbb{Q}[\mathbf{x}_\infty]$ and Λ in an algebraically independent manner.
- Representations inside $\tilde{\Lambda}$ are the same for $S_{\mathbb{N}}$ and for S_∞.

Stable Representations

For $\lambda \vdash n$ and $d \geqslant 0$, write $\mathrm{SSYT}_d(\lambda)$ for the set of those $M \in \mathrm{SSYT}(\lambda)$ whose entry sum is d.

Theorem

For every n and d, a decomposition of $\mathbb{Q}[\mathbf{x}_n]_d$ into irreducibles as a representation of S_n is given by $\bigoplus_{\lambda \vdash n} \bigoplus_{M \in \mathrm{SSYT}_d(\lambda)} V_M$.

Proposition

If $n > 2d$ then $\hat{\imath} : \bigcup_{\lambda \vdash n} \mathrm{SSYT}_d(\lambda) \to \bigcup_{\mu \vdash n+1} \mathrm{SSYT}_d(\mu)$ is a bijection.

Combined with the theorem, this presents $\left\{\mathbb{Q}[\mathbf{x}_n]_d\right\}_n$ as a *stable representation*, also known as a *finitely generated FI-module*.
This is the "same $\forall n \geqslant 2$" assertion for $d = 1$ from above.

Infinite Tableaux

Definition

- An *infinite Ferrers diagram* is a Ferrers diagram in which the first row is infinite. We denote that by $\hat{\lambda} \vdash \infty$.
- An element $\hat{T}$ of $\mathrm{SYT}(\hat{\lambda})$ is a filling of $\hat{\lambda}$ in a standard manner.
- We write $\hat{M} \in \mathrm{SSYT}(\hat{\lambda})$ for a semi-standard filling of $\hat{\lambda}$, with an indication of the finitely many non-zero entries at the "end" of the first, infinite line.

$$\hat{T} := \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 7 & 9 & \cdots \\ \hline 3 & 5 & 8 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}, \qquad \hat{M} := \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 1 & 2 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array},$$

where in $\hat{M}$ we also consider a 2 and a 5 at the "end" of the infinite row.

Theorem

- For $\hat{\lambda} \vdash \infty$, $\hat{T} \in \mathrm{SYT}(\hat{\lambda})$, and $\hat{M} \in \mathrm{SSYT}(\hat{\lambda})$ there is a construction of an element $F_{\hat{M},\hat{T}} \in \tilde{\Lambda}$ as the image of a monomial $p_{\hat{M},\hat{T}}$ under a "Young symmetrizer $\varepsilon_{\hat{T}}$ modulo stabilizer".
- This construction produces precisely the inverse limits of $\{F_{\hat{\iota}M,\iota T} \to F_{M,T}\}$.
- The space $V_{\hat{M}}$ spanned by $\{F_{\hat{M},\hat{T}} \mid \hat{T} \in \mathrm{SYT}(\hat{\lambda})\}$, as a basis, is an irreducible representation of both $S_{\mathbb{N}}$ and S_{∞}.
- The isomorphism classes of irreducible representations obtained in this way is in bijection with infinite Ferrers diagrams $\hat{\lambda} \vdash \infty$.

Limits of Stable Representations

We let $\mathrm{SSYT}_d(\hat{\lambda})$ consist of those $\hat{M} \in \mathrm{SSYT}(\hat{\lambda})$ whose entry sum (including the entries at the "end" of the infinite row) is d.
We can now present the direct limit of $\left\{\mathbb{Q}[\mathbf{x}_n]_d\right\}_n$.

Theorem

The sum $\sum_{\hat{\lambda}\vdash\infty}\sum_{\hat{M}\in\mathrm{SSYT}(\hat{\lambda})} V_{\hat{M}}$ is finite and direct, and produces the maximal completely reducible sub-representation $\tilde{\Lambda}_d^0$ inside $\tilde{\Lambda}_d$.

For $d = 1$ we have $\tilde{\Lambda}_1^0 = \mathbb{Q}[\mathbf{x}_\infty]_1^0 \oplus \mathbb{Q}e_1$ (see above).
There is also a conjectural finite filtration, with explicit maximal completely reducible sub-quotients, yielding the full representation $\tilde{\Lambda}_d$.
Intersecting this filtration with $\mathbb{Q}[\mathbf{x}_\infty]_d$ presents it, or its semi-simplification, as the inverse limit of this stable representation.

https://arxiv.org/abs/2505.07097
https://arxiv.org/abs/2505.07098
https://arxiv.org/abs/2505.07099

Thank you very much for your attention!

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