Partial automorphisms of combinatorial structures

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Combinatorial structure - classical approach to symmetries

- For a combinatorial structure $\mathcal{C} = (V, \mathcal{F})$

  Examples: graphs, directed graphs, hypergraphs, geometries, designs, ...

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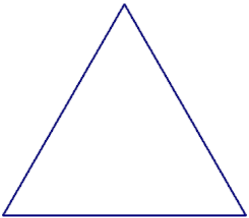
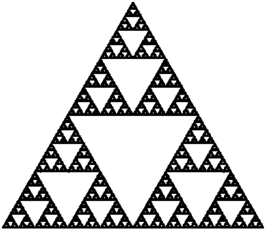
- symmetries - automorphisms of $(V, \mathcal{F})$

- Automorphisms form a group, $Aut(\mathcal{C}) \leq Sym(V)$.

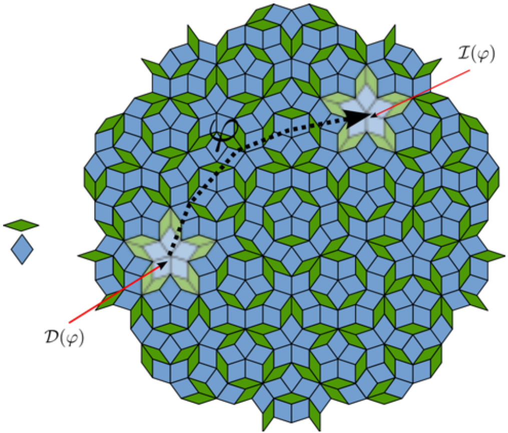
Partial Symmetries





Partial Symmetries

Penrose tiling



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- in particular, $Aut(\Gamma)$ is trivial for all those graphs

Partial Automorphisms

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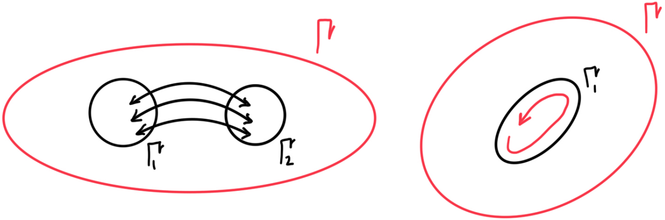

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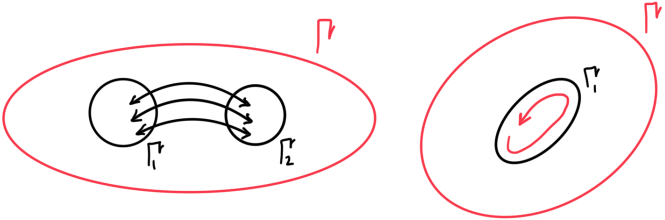

A partial automorphism of a combinatorial structure $\mathcal{C}$ is an isomorphism between two *induced* substructures.

Inverse Monoid of Partial Automorphisms

The set of all partial automorphisms, denoted $\mathrm{PAut}(\Gamma)$ with the composition and partial inverse of partial maps forms an inverse monoid.

$\mathrm{PAut}(\Gamma) \leq \mathrm{PSym}(V)$

A rank of a partial automorphism is given by the size of its domain.

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- No finite graph of order greater than 1 has a trivial inverse monoid of partial automorphisms $\mathrm{PAut}(\Gamma)$
- The inverse monoid $\mathrm{PAut}(\Gamma)$ is a complete algebraic description of Γ
- $PAut(\Gamma)$ contains $Aut(\Gamma)$ as its subgroup

Wagner-Preston representation

While groups can be represented as symmetries:

Theorem (Cayley)

Every group can be embedded in the set of one to one transformations on a set.

Inverse semigroups can be represented as partial symmetries:

Theorem (Wagner-Preston)

Every inverse semigroup can be embedded in the set of partial one to one transformations on a set.

Problem

Given a class of combinatorial structures and an inverse monoid M, is there a combinatorial structure $\mathcal{C}$ from our class whose full partial automorphism monoid $\mathrm{PAut}(\mathcal{C})$ is isomorphic to M?

Analogue of Frucht's theorem for groups.

Classification Problem

Given a class of combinatorial structures, classify finite inverse monoids M for which there exists a structure $\mathcal{C}$ from the considered class whose full partial automorphism monoid $\mathrm{PAut}(\mathcal{C})$ is isomorphic to M.

Analogue of GRR's for groups.

Structure of PAut(Γ) for graph (digraph, colored graph, multigraph,...) Γ

Proposition (R.Jajcay,T.Jajcayova,N.Szakács,M.Szendrei 2021)

For any graph Γ, *the* $\mathcal{D}$*-classes of* $\mathrm{PAut}(\Gamma)$ *correspond to the isomorphism classes of induced subgraphs of* Γ, *that is, two elements are* $\mathcal{D}$*-related if and only if the subgraphs induced by their respective domains (or images) are isomorphic.*

Partial order for $\mathcal{D}$-classes: "subgraph" relation

Example

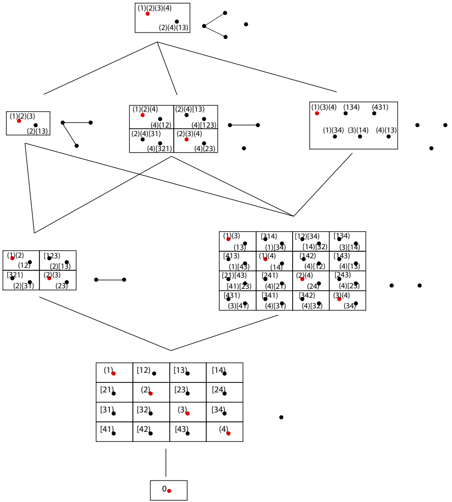

Figure: The Green-class structure of partial graph automorphisms

When is an inverse monoid of partial permutations the partial automorphism monoid of a graph?

Theorem (R.Jajcay,T.Jajcayova,N.Szakács,M.Szendrei 2021)

Given an inverse submonoid $S \leq \mathrm{PSym}(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S if and only if the following conditions hold:

1. *S is a full inverse submonoid of $\mathrm{PSym}(X)$,*
2. *for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,*
3. *the rank 2 elements of S form at most two $\mathcal{D}$-classes,*
4. *the $\mathcal{H}$-classes of rank 2 elements are nontrivial.*

When is an (abstract) inverse monoid *isomorphic* to the partial automorphism monoid of a graph?

Theorem (R.Jajcay,T.Jajcayova, N.Szakács, M.Szendrei 2021)

Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S if and only if the following conditions hold:

1. *S is Boolean,*
2. *S is fundamental,*
3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*
4. *S has at most two $\mathcal{D}$-classes of height 2,*
5. *the $\mathcal{H}$-classes of the height 2 $\mathcal{D}$-classes of S are nontrivial.*

Symmetry Level of a Graph

Definition

The **level of symmetry** of a graph Γ of order n is the ratio between the largest rank of a non-trivial partial automorphism of Γ and its order n.

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The **level of symmetry** of a graph Γ of order n is the ratio between the largest rank of a non-trivial partial automorphism of Γ and its order n.

- A graph Γ admitting a non-trivial automorphism has level of symmetry 1
- The level of symmetry of Γ is equal to the level of symmetry of its complement

Symmetry Level of a Graph

Question 1: Given $k \geq 1$, does there exist a graph Γ of order n with level of symmetry equal to $\frac{n-k}{n}$?

Computational results (J.Pastorek, V.Cingel 2023+):

- through exhaustive search: for $n \leq 10$ all graphs have the level of symmetry at least $\frac{n-1}{n}$.
- he has complete list of graphs on $n = 11$ vertices with the level of symmetry $\frac{n-2}{n}$ and there are no graphs with the level of symmetry $\frac{n-3}{n}$ or less, for $n = 11$.
- through constructions: constructed graphs with the smaller level of symmetry; his record is $\frac{n-6}{n}$, for $n = 27$.

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Question 2: What is the minimal level of symmetry of a graph Γ of order n as a function of n?

Connection to Isomorphism Problem

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- Computationally, one of the big problems in determining $Aut(\Gamma)$ is the part where one needs to prove that the list of automorphisms already found is complete.
- Result by Rudi Mathon: Dermining orbits of $Aut(\Gamma)$ of a graph is polynomially equivalent to determining $Aut(\Gamma)$
- Well-known Weisfeiler-Leman (W-L and k-W-L) Algorithm that aproximate orbits $Aut(\Gamma)$ of a graph. Stable coloring can be computed in time $O(n^k \log n)$

Connection to Isomorphism Problem

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- Given an inverse monoid of partial automorphisms of Γ, if one wishes to determine whether it is the complete $PAut(\Gamma)$, one should ask whether it satisfies the "JJSS" characterization
- A partial automorphism φ of Γ is **complete** if there exists no partial automorphism ψ of Γ that is an extension of φ
- The level of symmetry of Γ of order n is the ratio between the biggest rank of a *complete* partial automorphism of Γ and n

Algorithmic aspects of determining $PAut(\Gamma)$

- A partial automorphism φ of Γ is complete if and only if there exists **no** pair of vertices u, v, $u \notin dom\varphi$ and $v \notin ran\varphi$ such that $u \sim w$ if and only $v \sim \varphi(w)$, for all $w \in dom\varphi$.

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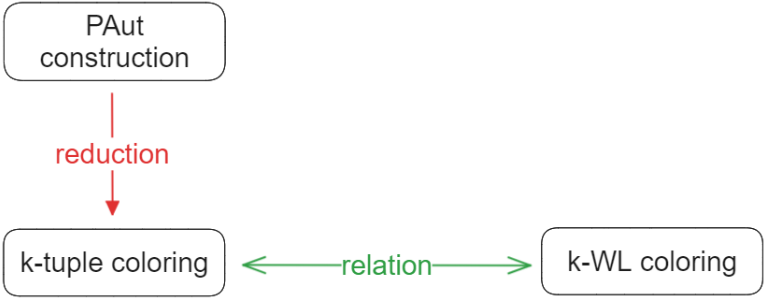
- If a partial automorphism φ of Γ is not complete due to a pair u, v, the partial automorphism $\tilde{\varphi} : dom\varphi \cup \{u\} \to ran\varphi \cup \{v\}$ is an extension of φ
- Determining whether a partial automorphism is complete and forming all of its extensions of rank increased by 1 is polynomial in the order of Γ

Comparison of the Two Algorithms

- The algorithm for constructing $PAut(\Gamma)$ can be adjusted to determine the orbits of $Aut(\Gamma)$ as follows:
  The vertices u, v receive different colors at level k if no partial isomorphism of rank k can be extended by adding $u \mapsto v$
- We are working on comparing the complexity of the two algorithms, and its performance on various graph classes.


PAut construction
reduction
k-tuple coloring
relation
k-WL coloring

Thank you for listening!