Highly symmetric Steiner and Kirkman triple systems

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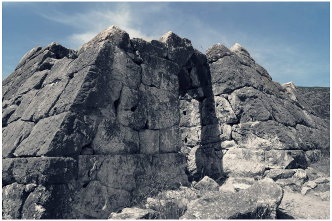

5th Pythagorean conference
June 4, 2025

Steiner triple systems

A Steiner triple system STS(v) of order v is a pair $(V, \mathcal{T})$ where

- V is a set of v points (usually, $V = [1, v] := \{1, \ldots, v\}$)
- $\mathcal{T}$ is a set of triples of V

such that any two points lie in exactly one triple of $\mathcal{T}$

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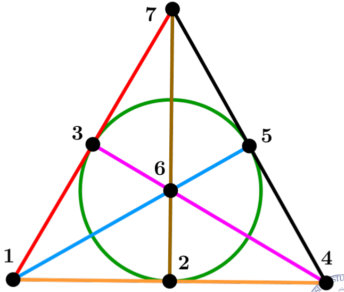
Here is an STS(7) where

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\mathcal{T} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$$



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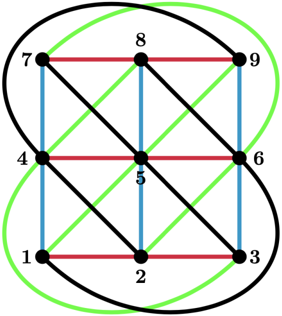
Here is an STS(9) where

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\mathcal{T} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \{1,5,9\}, \{2,6,7\}, \{3,4,8\}, \{1,6,8\}, \{2,4,9\}, \{3,5,7\}\}$$



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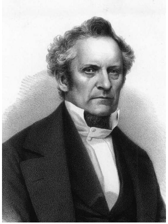

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- Plücker 1839: necessarily, v is odd and $|\mathcal{T}| = \frac{v(v-1)}{6}$

$$\Updownarrow$$

$$v \equiv 1, 3 \pmod{6}$$

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- Plücker 1839: necessarily, $v \equiv 1, 3 \pmod 6$

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There is an STS(v) IFF $v \equiv 1, 3 \pmod 6$

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- Kirkman's proof employs two recursive constructions that build STS($2v+1$) and STS($2v-5$) from an STS(v)
- It is remarkable that STSs for every admissible order are built starting only with the trivial system on one element

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Therefore, a systematic classification of these systems became of critical importance

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Consider two $STS(v)$, say $\mathbb{S} = (V, \mathcal{T})$ and $\mathbb{S}' = (V, \mathcal{T}')$

An isomorphism from $\mathbb{S}$ to $\mathbb{S}'$ is a permutation α of V s.t.

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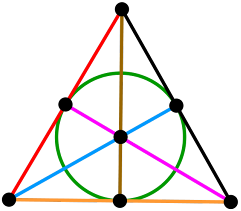
Up to isomorphism, there is exactly one $STS(7)$, which coincides with the point-line incidence structure of the projective plane PG(2, 2) over GF(2)



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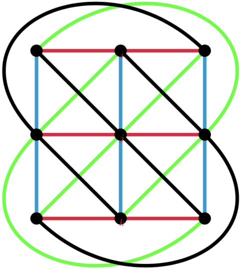

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- there is one $STS(7)$ and one $STS(9)$
- there are two $STS(13)$ [De Pasquale 1899, Brunel 1902]
- there are 80 $STS(15)$ [Cole, Cummings, White 1917-19]
- there are 11,084,874,829 $STS(19)$ [Kaski, Östergård 2004]
- there are 14,796,207,517,873,771 $STS(21)$ [Heinlein, Östergård 2024]

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- $\mathrm{Aut}(STS(7)) = PGL(2, 2) = GL(3, 2)$
- $\mathrm{Aut}(STS(9)) = AGL(2, 3) = (3^2) \times GL(2, 3)$

Kirkman triple systems

Let $\mathbb{S} = (V, \mathcal{T})$ be an STS(v), with $v \equiv 1, 3 \pmod 6$

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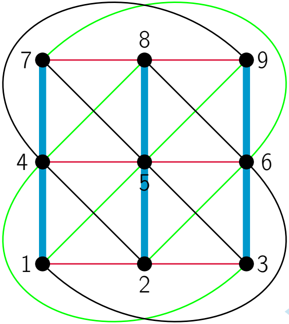

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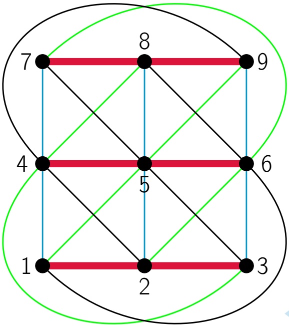

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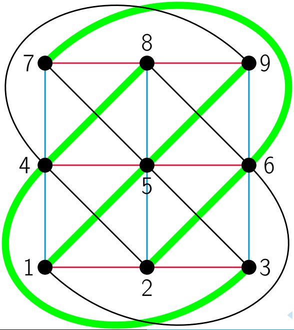

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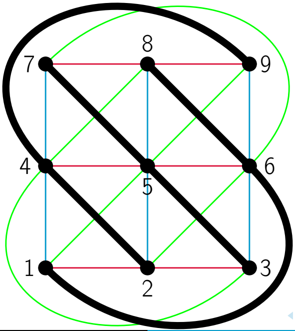

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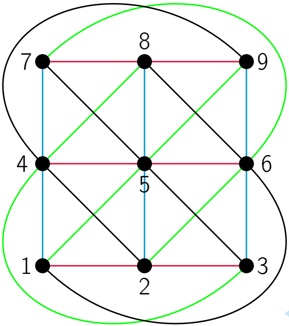

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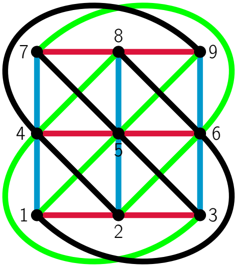
- $|\mathcal{P}| = \frac{|\mathcal{T}|}{v/3} = \frac{v(v-1)/6}{v/3} = \frac{v-1}{2}$



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The case $v = 15$, known as the "Kirkman schoolgirl problem", was first solved by Cayley in 1850

A different solution was given by Kirkman in 1851

	a	b	c	d	e	f	g
abc				35	17	82	64
ade		62	84			15	37
afg		13	57	86	42		
bdf	47		16		38		25
bge	58		23	14		67	
cdg	12	78			56	34	
cef	36	45		27			18

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Theorem (Lu Jiaxi 1960s, Ray-Chaudhuri, Wilson 1971)
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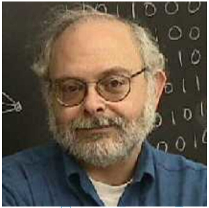

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Up to isomorphism

- there is exactly one KTS(9) and Aut$(KTS(9)) = AGL(2,3)$
- there are 7 KTS(15); but together they yield 4 nonisomorphic STS(15)s

Steiner and Kirkman triple systems

- Triple systems (Oxford Math Monographs) [Colbourn, Rosa 1999]

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- 3-pyramidal Steiner triple systems [Buratti, Rinaldi, Traetta 2017]
- Direct constructions of large sets of Kirkman triple systems [Zheng, Chang, Zhou 2017]
- KTS(n) with Minimum Block Sum Equal to n, for Access Balancing in Distributed Storage [Brummond 2019]
- Access balancing in storage systems by labeling partial Steiner systems [Meng Chee, Colbourn, Dau, Gabrys, Ling, Lusi, Milenkovic 2020]
- The first families of highly symmetric KTSs whose orders fill a congruence class [Bonvicini, Buratti, Garonzi, Rinaldi, Traetta 2021]
- Novàk's conjecture on cyclic STSs and its generalization [Feng, Horsley, Wang 2021]
- Automorphism groups of Steiner triple systems [Doyen, Kantor 2022]
- The spectrum of resolvable Bose triple systems [Lusi, Colbourn 2023]
- Weak colourings of Kirkman triple systems [Burgess, Cavenagh, Danziger, Pike 2025]

f-pyramidal STSs and KTSs

Let $\mathbb{B} = (V, \mathcal{B})$ be an STS(v) or a KTS(v)

Assume that $V = \{\infty_1, \ldots, \infty_f\} \cup [1, v - f]$

Let $G \leq Aut(\mathbb{B})$ be an automorphism group of $\mathbb{B}$

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$\mathbb{B}$ is called f-pyramidal under G if

- $\gamma(\infty_i) = \infty_i$, $\forall i \in [1, f]$ and $\gamma \in G$
- $\forall j, k \in [1, v - f]$ there is exactly one $\gamma \in G$ s.t. $\gamma(j) = k$ (that is, G acts sharply transitively on $[1, v - f]$)

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- Mazzuoccolo and Rinaldi (2007) considered f-pyramidal 1-factorizations

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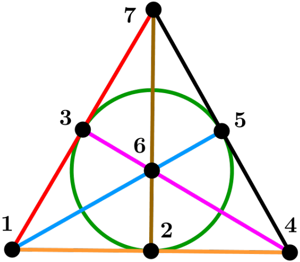
The unique STS(7) is sharply transitive over $G \simeq \mathbb{Z}_7$



$V = \mathbb{Z}_7$

$G = \{\tau_g \mid g \in \mathbb{Z}_7\} \simeq \mathbb{Z}_7$
where $\tau_g(x) = x + g$, for every $x \in G$

G is an automorphism group of STS(7) acting sharply transitively on the point-set V

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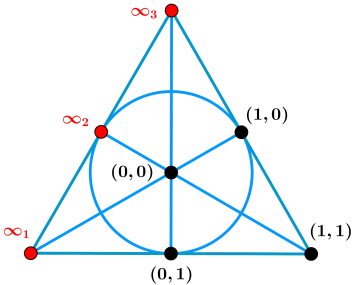
The unique STS(7) is 3-pyramidal over $\overline{G} \simeq G$



$$V = \{\infty_1, \infty_2, \infty_3\} \cup \overbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}^{G}$$

$\overline{G} = \{\tau_g \mid g \in G\} \simeq G$ where

$$\tau_g(x) = \begin{cases} x + g & \text{if } x \in G, \\ x & \text{if } x \in \{\infty_1, \infty_2, \infty_3\} \end{cases}$$

$\overline{G}$ is an automorphism group of STS(7) having a 3-pyramidal action on V

f-pyramidal STSs and KTSs

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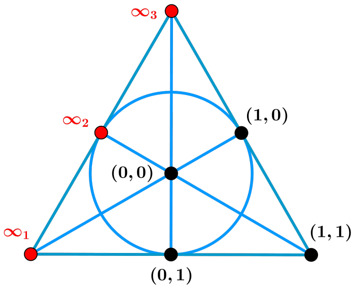

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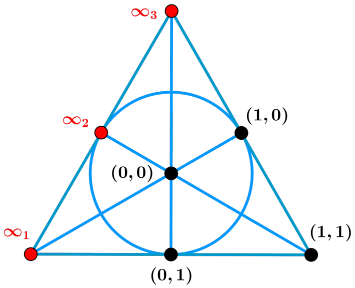

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Given an f-pyramidal STS(v), the set of f fixed points forms an STS(f) $\implies$ $f = 0$, or $f \equiv 1, 3 \pmod 6$ and $f < \frac{v}{2}$

The most studied cases: $f = 0, 1, 3$

- 0-pyramidal = sharply transitive
- 1-pyramidal = 1-rotational

A characterization of f-pyramidal STSs

Let $(G, +)$ be an additive group

- The list of differences of $F \subseteq G$ is the multiset

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- The list of differences of a family $\mathcal{F}$ of subsets of G is the multiset $\Delta\mathcal{F} = \bigcup_{F \in \mathcal{F}} \Delta F$

Let $\mathcal{F} = \{F_1, F_2\}$. Then

$$\Delta\mathcal{F} = \Delta F_1 \cup \Delta F_2 = \mathbb{Z}_{15} \setminus \{0, \pm 6\}$$

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$\Sigma = \{0\}$ is a partial spread of $(G, +)$ of type $(0, 0)$

$\Sigma = \{0, 6, \pm 4\}$ is a partial spread of $\mathbb{Z}_{12}$ of type $(1, 2)$

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Theorem[1]**.** There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, if and only if

1. $|G| = v - f$
2. G has exactly f involutions, and
3. there exists a $(G, \Sigma, 3)$-DF, where Σ is a partial spread of G of type $(f, 2e)$

[1] M. Buratti, G. Rinaldi, TT, *Ars Math. Contemp.* 13 (2017)

A characterization of f-pyramidal STSs

Theorem. (Buratti, Rinaldi, TT 2017) There exists an f-pyramidal STS(v) under G, with $f < \frac{v}{2}$, IFF $|G| = v - f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3)$-DF where Σ is a partial spread of G of type $(f, 2e)$

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$\Rightarrow$ $\mathcal{F}$ is a $(G, \Sigma, 3)$-DF $\Rightarrow$ $\exists$ a 3-pyramidal STS(15)

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$$\Downarrow$$

Letting $\Sigma^+ := \{\{\infty_1,(0,0,0),s_1\},\{\infty_2,(0,0,0),s_2\},$
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then, the set $\mathcal{T}$ of all the distinct translates of triples in

$$\mathcal{F} \,\cup\, \Sigma^+ \,\cup\, \Big\{\{\infty_1,\infty_2,\infty_3\}\Big\}$$

is a 3-pyramidal STS(15)

The spectrum problem for f-pyramidal TSs

Let $f = 0$ or $f \equiv 1, 3 \pmod 6$. Determine all v for which there exists an f-pyramidal STS(v) (resp. KTS(v))

The case $f = 0$: sharply transitive STS(v)

- There is a sharply transitive STS(v) IFF $v \equiv 1, 3 \pmod 6$

  [Peltesohn 1939]

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- A doubly transitive STS is either $PG(n, 2)$ or $AG(n, 3)$ [Key, Shult, 1984 – Hall, 1985 – Kantor 1985]

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- There is a 1-rotational STS(v) over a group G, with G
  - cyclic IFF $v \equiv 3, 9 \pmod{24}$ [Phelps, Rosa, 1981]
  - abelian IFF $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$ [Buratti, 2001]
  - dicyclic IFF $v \equiv 9 \pmod{24}$ [Mishima, 2008]

1-rotational STS(v): open cases

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Theorem[1]**.** The existence of a 1–rotational STS(v) is undecided if simultaneously

a. $v - 1 = (p^3 - p)n \equiv 0 \pmod{96}$, with p prime and $n \not\equiv 0 \pmod{4}$

b. the odd part of $v - 1$ is square free and all prime divisors are $\not\equiv 1 \pmod{6}$

[1] Bonvicini, Buratti, Rinaldi, TT, *Des. Codes Cryptogr. 62 (2012)*

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The first two open cases:

n	p	v	Admissible groups
2	23	24289	"extension of $PGL_2(23)$ by $\mathbb{Z}_2$"
1	47	103777	$SL_2(47)$

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Undecided cases with n and p small:

n	p (< 1000)	Admissible groups
1 $\equiv_4 1, 3$	$47, 353, 383, 479, 641...$	$SL_2(p)$ .. $SL_2(p) \times \mathbb{Z}_n$ ..
2 $\equiv_4 2$	$23, 47, 137, 263, 353, 383, 479, 641, 983...$	G .. $G \times \mathbb{Z}_{\frac{n}{2}}$ ..

G = "extension of $PGL_2(p)$ by $\mathbb{Z}_2$"

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Theorem[1]. There is a 3-pyramidal STS(v) IFF
$v \equiv 7, 9, 15 \pmod{24}$ or $7 \leq v \equiv 3, 19 \pmod{48}$

v	Existence	Group
$24n + 3$	Yes $\Longleftrightarrow$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n + 7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n + 9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n + 15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n + 19$	Yes $\Longleftrightarrow$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$

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$24n+7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n+9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n+15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n+19$	Yes $\iff$ n is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$

- There is an abelian 3-pyramidal STS(v) IFF
  $v \equiv 7, 15 \pmod{24}$ or $7 \leq v \equiv 3, 19 \pmod{48}$

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f–pyramidal STSs over abelian groups

There exists an f–pyramidal STS(v), with $0 \leq f \leq 3$, over some abelian group IFF

- $f = 0$ and $v \equiv 1, 3 \pmod{6}$ [Peltesohn 1939]
- $f = 1$ and $v \equiv 1, 3, 9, 19, 27, 33, 51, 57 \pmod{72}$ [Phelps, Rosa 1981 - Buratti 2001]
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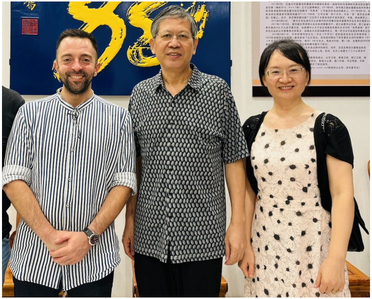

Theorem. (Chang, TT, Zhou, 2025+)

There exists an f–pyramidal STS(v), with $3 < f < \frac{v}{2}$, over some abelian group IFF $f = 2^m - 1$, for some $m \geq 3$, and

- $v \equiv 2^{m+1} - 1 \pmod{2^m 3}$, or
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Proof. ($\Rightarrow$) Assume there is an f−pyramidal STS(v) ($f > 3$) over an abelian group G.

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Theorem[1]**.** There is an f-pyramidal STS(v) under G IFF $v = |G| + f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3)$-DF where Σ is a PS of type $(f, 2e)$

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Proof. ($\Rightarrow$) Assume there is an f–pyramidal STS(v) ($f > 3$) over an abelian group G. Necessarily,

$v = |G| + f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3)$-DF, say $\mathcal{F}$, where Σ is a PS of type $(f, 2e)$

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$v = |G| + f$, G has exactly f involutions, and there exists a $(G, \Sigma, 3)$-DF, say $\mathcal{F}$, where Σ is a PS of type $(f, 2e)$

Then, $f + 1 = 2^m$ $(m \geq 3)$,

f–pyramidal STSs over abelian groups

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- If $d \not\equiv 0 \pmod 3$, then $e = 0$ and $2^{m-1}(d-1) \equiv 0 \pmod 3$, that is, $d \equiv 1 \pmod 3$, say $d = 3n + 1$.

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Theorem. (Chang, TT, Zhou, 2025+)
There exists an f–pyramidal STS(v), with $3 < f < \frac{v}{2}$, over some abelian group IFF $f = 2^m - 1$, for some $m \geq 3$, and

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- If $d \not\equiv 0 \pmod 3$, then $d \equiv 1 \pmod 3$, say $d = 3n + 1$, hence $v = 2^m 3n + 2^{m+1} - 1 \Rightarrow$ (1)
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$$G = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$$

of order $|G| = v - f = 160$

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- We need a $(G, \Sigma, 3)$-DF, say $\mathcal{F}$, where Σ is a PS of type $(15, 0)$.

In other words, $\Sigma = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times 10\mathbb{Z}_{20}$

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- For every $S \subseteq G$, set $S^* = S \setminus \Sigma$. Hence, $\Delta\mathcal{F} = G^*$

Since $G^* = |G \setminus \Sigma| = 160 - 16 = 144$, then $|\mathcal{F}| = \frac{144}{6} = 24$.

A 15–pyramidal STS(175) over G

Proof. ($\Leftarrow$) $G = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\Sigma = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times 10\mathbb{Z}_{20}$

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- Set $\mathbb{Z}_2^2 = \{0, \alpha, \beta, \gamma\}$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_{20}$. Hence, $G = \mathbb{Z}_2^2 \times H$

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Note that $G^* = \bigcup_{x \in \mathbb{Z}_2^2} \{x\} \times H^*$, where $|H^*| = 36$

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$$\mathcal{T}' = \{\ldots\ T' = \{(0, *, *), (\alpha, *, *), (\gamma, *, *)\} \ldots\}$$

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Then, $\Delta\mathcal{T} = \left(\{\alpha, \beta, \gamma\} \times H^*\right) \cup \pm\{(0,1,2), (0,1,4), (0,1,6)\}$

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Proof. $(\Leftarrow)$ $G = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\Sigma = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times 10\mathbb{Z}_{20}$
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- It is left to build a set $\mathcal{W} = \{W_1, \ldots, W_5\}$ of 5 triples such that

$$\Delta\mathcal{W} = \left(\{0\} \times H^*\right) \setminus \pm\{(0,1,2),(0,1,4),(0,1,6)\}$$

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$$\Delta\mathcal{W} = \left(\{0\} \times H^*\right) \setminus \pm\{(0,1,2), (0,1,4), (0,1,6)\} = \Delta_1 \cup \Delta_2$$

where $\Delta_1 = \{(0,0)\} \times \mathbb{Z}_{20}^*$ and $\Delta_2 = \{(0,1)\} \times (\mathbb{Z}_{20}^* \setminus \pm\{2,4,6\})$

A 15–pyramidal STS(175) over G

Proof. $(\Leftarrow)$ $G = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\Sigma = \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times 10\mathbb{Z}_{20}$

We need a $(G, \Sigma, 3)$-DF, say $\mathcal{F}$, to build a 15-pyramidal STS(175) over G. Hence, $|\mathcal{F}| = 24$ and $\Delta\mathcal{F} = G^* = G \setminus \Sigma$

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$$\Delta W_i = \begin{cases} \{(0,0,*)^6\} & \text{if } i = 1,2, \\ \{(0,0,*)^2, (0,1,*)^4\} & \text{if } i = 3,4,5. \end{cases}$$

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- $\mathcal{F} = \mathcal{T} \cup \mathcal{W}$ is the desired $(G, \Sigma, 3)$-DF
- To fill the $*$ we might need Langford sequences

The spectrum problem for f-pyramidal KTSs

The case $f = 0$: sharply transitive KTS(v)

- A necessary condition for a cyclic KTS($6n+3$) to exist:

$$2n + 1 \text{ is not a prime power} \equiv 5 \pmod 6$$

- There is a cyclic KTS($6n+3$) whenever:
  - each prime factor of $2n+1$ is $\equiv 1 \pmod 6$ [Genma, Mishima, Jimbo 1997]
  - $6n + 3 < 200$ [Meszka, Rosa 2007]

STS

The spectrum problem for f-pyramidal KTSs

The case $f = 1$: 1-rotational KTS(v)

- $AG(n, 3)$ is a 1-rotational KTS(3^n)
- There is a 1-rotational KTS($2n + 1$) whenever each prime factor of n is $\equiv 1 \pmod{12}$ [Buratti 1998]
- Up to isomorphisms, there are exactly 500 1-rotational KTS(33) [Buratti, Zuanni 2000]
- There is a 1-rotational KTS($8n + 1$) whenever each prime factor of n is $\equiv 1 \pmod{6}$ [Buratti, Zuanni 2001]

The spectrum problem for 3–pyramidal KTSs

Admissible orders: $v \equiv_{24} 9, 15$ or $v \equiv 4^m \cdot 48 + 3 \pmod{4^m \cdot 96}$

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Theorem[1]**.** There is a 3-pyramidal KTS(v) over G whenever

- $v = 24n + 9 = 6(4n + 1) + 3$

  $4n + 1 = q_1 q_2 \cdots q_t$ and each q_i is a prime power $\equiv 1 \pmod 4$
- $v = 24n + 15 = 12(2n + 1) + 3, \quad 2n + 1 = q_1 q_2 \cdots q_t,$ and

  $3 \mid 2n + 1$, or the square free part of $2n + 1$ has no prime $\equiv_{12} 11$
- $v \equiv 4^m \cdot 48 + 3 \pmod{4^m \cdot 96}$




[1] Bonvicini, Buratti, Garonzi, Rinaldi, TT, *Des. Codes Cryptogr. 89 (2021)*

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  $G = Alt(4) \times \mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_t}$ First open case: KTS($135 = 12 \cdot 11 + 3$)

- $v \equiv 4^m \cdot 48 + 3 \pmod{4^m \cdot 96}$

  $G = (\mathbb{Z}_4^{m+2} \ltimes \mathbb{Z}_3) \times \mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_t}$ where $q_1 \cdots q_t = \frac{v-3}{4^m \cdot 48}$

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Corollary[1]**.**

There exists a KTS(v) with at least $v - 3$ automorphisms whenever

$v \equiv 39 \pmod{72}$ *or* $v \equiv 4^m \cdot 48 + 3 \pmod{4^m \cdot 96}$

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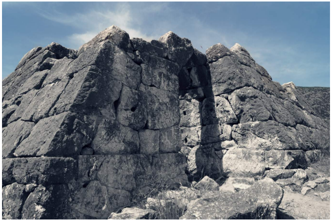
Highly symmetric Steiner and Kirkman triple systems

Tommaso Traetta
University of Brescia, Italy



5th Pythagorean conference
June 4, 2025