A family of strongly regular graphs from hyperbolic quadrics

Valentino Smaldore

Università degli Studi di Padova

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joint work with A. Cossidente, J. De Beule, G. Marino and F. Pavese

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Spectral graph theory
Definitions

$$G := (V(G), E(G))$$

$V = V(G)$ is a non-empty set, of element called *vertices*
$E = E(G)$ is the set of *edges*, together with an *incidence function* $\phi : E \to V \times V$. If $\phi(e) = \{u, v\}$ we say that e *joins* u and v, and those are called *adjacent vertices* or *neighbours*.

Definition

The adjacency matrix A of a graph G of v vertices is a symmetric $v \times v$ matrix in which the coordinate $A(i,j)$ is the number of edges between v_i and v_j.

Remark: if we do not allow *loops* and *multiedges*, the adjacency matrix is a $(0,1)$-matrix, and $A(i,i) = 0$, $i = \{1, \dots, v\}$.

Spectral graph theory

Eigenvalues

Definition

The spectrum $(\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_n^{m_n})$ *of a matrix is the set of all its eigenvalues* θ_i*, counted with their respective multiplicities* m_i*.*

Spectral graph theory studies a graph, taking as point of view the spectrum of its adjacency matrix.

Proposition

The spectrum of the adjacency matrix is a graph invariant.
Two graphs are called cospectral if the adjacency matrices have equal multisets of eigenvalues.

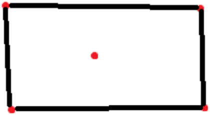
Spectral graph theory

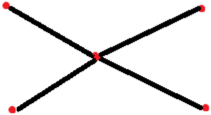
Cospectral graphs

Proposition

Cospectral graphs need not to be isomorphic, but isomorphic graphs are always cospectral.

Both $C_4 \cup K_1$ and S_4 have spectrum $(-2, 0^3, 2)$.





Spectral graph theory

Strongly regular graphs

Definition

A strongly regular graph with parameters (v, k, λ, μ) is a graph with v vertices, each vertex lies on k edges, any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours.

Theorem

A strongly regular has exactly three eigenvalues: k, θ_1 and θ_2 of multiplicity, respectively, 1, m_1 and m_2, where k is the regularity and:

$$\theta_{1,2} = \frac{1}{2}\left[(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right],$$

$$m_{1,2} = \frac{1}{2}\left[(v - 1) \mp \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right].$$

Graph $NO^{+}(2n+2,2)$

Consider the vector space $V(2n+2,2)$, together with a non-degenerate hyperbolic quadratic form.

Definition

$NO^{+}(2n+2,2)$ is the graph whose vertex set is the set of non-singular points, and two vertices are adjacent if they are orthogonal, i.e. their connecting line is tangent to the hyperbolic quadric $Q^{+}(2n+1,2)$, defined by the quadratic form.

Graph $NO^+(2n+2,2)$

Spectrum of $NO^+(2n+2,2)$

Proposition

$NO^+(2n+2,2)$ is a strongly regular graph with parameters:
$v = 2^{2n+1} - 2^n$,
$k = 2^{2n} - 1$,
$\lambda = 2^{2n-1} - 2$,
$\mu = 2^{2n-1} + 2^{n-1}$.
The spectrum is
$(2^{2n} - 1, -(2^n+1)^{\frac{1}{3}(2^n-1)(2^{n+1}-1)}, (2^{n-1}-1)^{\frac{4}{3}(2^{2n}-1)})$.

Example

$NO^+(8,2)$ is an srg$(120,63,30,36)$ with spectrum $(63, -9^{35}, 3^{84})$.

Graph $NO^+(2n+2,2)$

A quadric with a hole

A.E. Brouwer, A.V. Ivanov, M.H. Klin, *Some new strongly regular graphs*, Combinatorica, 1989, 9(4), 339-344.

Consider the graph Δ with point set $X=Q\setminus M$, where two points x, y are adjacent when the projective line xy is contained in X. (Thus, Δ is a partial subgraph, not an induced subgraph, of Γ.)

Define

$$\theta_j = \frac{q^j-1}{q-1}.$$

Theorem 1. *Δ is strongly regular with parameters $v=q^{m-1}\theta_m$, $k=q^{m-1}\theta_{m-1}$, $\lambda=q^{m-1}\theta_{m-2}+q^{m-2}(q-1)$, $\mu=q^{m-1}\theta_{m-2}$, $r=q^{m-1}$, $s=-q^{m-2}$. It has automorphism group $q^{m(m-1)/2}L_m(q)$, acting (imprimitively) rank 4 (for $m\geqq 3$, $q>2$ or $m\geqq 4$, $q=2$). The following three conditions are equivalent:* (i) *Δ is pairwise 4-regular,* (ii) *Δ is a graph in the switching class of a regular two-graph, and* (iii) $q=2$.

Graph $\mathcal{G}_n$

Consider the quadric $Q^+(2n+1,q)$ with $\perp$ its induced polarity of $PG(2n+1,q)$. Fix a generator Π of $Q^+(2n+1,q)$.

$V(\mathcal{G}_n) :=$ points of $Q^+(2n+1,q) \setminus \Pi$.

We define two relations on V. Let $P, Q \in V$, then $P \sim_1 Q$ if and only if the line $\langle P, Q\rangle$ is secant to $Q^+(2n+1,q)$, and $P \sim_2 Q$ if and only if $\langle P, Q\rangle$ is totally isotropic and meets the Π in a point.

Theorem

The graph $\mathcal{G}_n$ is a $srg(\frac{q^n(q^{n+1}-1)}{q-1}, q^{2n}-1, q^{2n-1}(q-1)-2, (q^{2n-1}+q^{n-1})(q-1))$.

Graph $\mathcal{G}_n$

Graph $\mathcal{G}_3$

Proposition

When $q = 2$, the graph $\mathcal{G}_n$ is a $srg(2^{2n+1} - 2^n, 2^{2n} - 1, 2^{2n-1} - 2, 2^{2n-1} + 2^{n-1})$. Hence, it is cospectral to $NO^+(2n+2, 2)$.

Theorem

The graphs $\mathcal{G}_3$ and $NO^+(8, 2)$ are not isomorphic.

The isomorphism issue
Classification of cliques

Definition

A clique of the graph Γ is a set of pairwise adjacent vertices. A clique is said to be maximal if it is maximal with respect to set theoretical inclusion.

Result (Delsarte clique bound)

Let Γ be a strongly regular graph with regularity k and smallest eigenvalue θ. Then the size of a clique in Γ is at most $1 - \frac{k}{\theta}$.

Corollary

The size of a clique in $NO^+(8,2)$ and $\mathcal{G}_3$ is at most 8.

The isomorphism issue

Classification of cliques

Proposition

- *The maximal cliques of* $NO^+(8,2)$ *consist of the 2025 3-spaces cutting the quadric* $Q^+(7,2)$ *in a plane.*
- $PGO^+(8,2)$ *has the following orbits on maximal cliques of* $\mathcal{G}_3$*:*

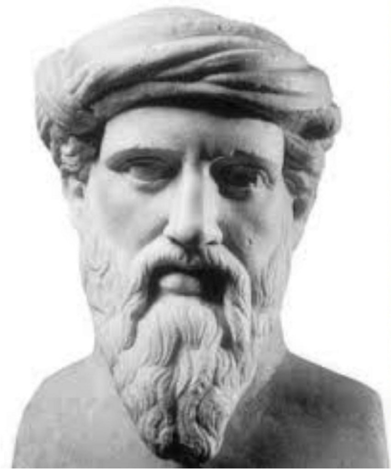
# cliques	Size	Adj.	Geometric description
10752	*5*	$\sim_1$	$Q^-(3,2) \subseteq Q^+(7,2)$ *not meeting* Π
960	*8*	$\sim_1$	*Cameron-Praeger ovoid*
15	*8*	$\sim_2$	*Generators of* $Q^+(7,2)$ *meeting* Π *in a plane*
840	*8*	*mixed*	*Cones* $PQ^-(3,2)$, $P \in \Pi$, *meeting* Π *in a line*
210	*8*	*mixed*	*Cones* $\ell Q^+(1,2)$, $\ell \in \Pi$, *meeting* Π *in* ℓ

Hence, $\mathcal{G}_3$ *have 10752 cliques of size 5 and 2025 cliques of size 8.*

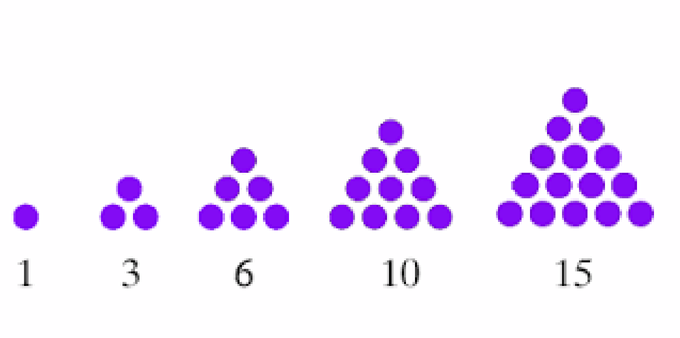
Fun facts

Fun facts

Pythagoras of Samos 570-495 B.C.

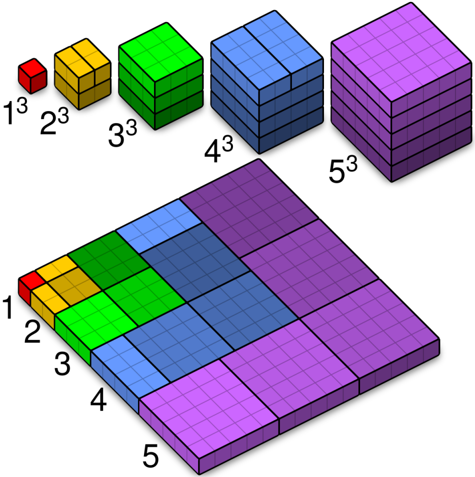


Fun facts
Triangular numbers



Fun facts

Squared triangular numbers



Fun facts

Squared triangular numbers

Theorem (Nicomachus's Theorem)

$$\left(\sum_{i=1}^{n} i\right)^2 = \left(\sum_{i=1}^{n} i^3\right)$$

$$(1+2+3+4+5+6+7+8+9)^2 = 1^3+2^3+3^3+4^3+5^3+6^3+7^3+8^3+9^3$$

$$45^2 = 2025$$

The isomorphism issue

Classification of cliques

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Hence, $\mathcal{G}_3$ *have 10752 cliques of size 5 and 2025 cliques of size 8.*

The isomorphism issue

Cliques of Type 1

Definition

- *An ovoid* $\mathcal{O}$ *of* $Q^{+}(7,2)$ *is a set of isotropic points such that every generator contains exactly one point of* $\mathcal{O}$.
- *A partial ovoid* $\mathcal{O}$ *of* $Q^{+}(7,2)$ *is a set of isotropic points such that every generator contains at most one point of* $\mathcal{O}$. *A partial ovoid is said to be maximal if it is maximal with respect to set-theoretic inclusion.*

Lemma

An ovoid of $Q^{+}(7,2)$ *gives rise to a maximal clique of size 8 of* $\mathcal{G}_3$.

The isomorphism issue

Cliques of Type 1

Lemma

A partial ovoid of $Q^{+}(7,2)$ gives rise to a clique of Type 1.

Lemma

An elliptic quadric $Q^{-}(3,2)$ contained in $Q^{+}(7,2)$ is a maximal partial ovoid.

Lemma

An elliptic quadric $Q^{-}(3,2)$ not touching Π gives rise to a maximal clique of size 5 of $\mathcal{G}_3$.

The isomorphism issue

Classification of cliques

Proposition

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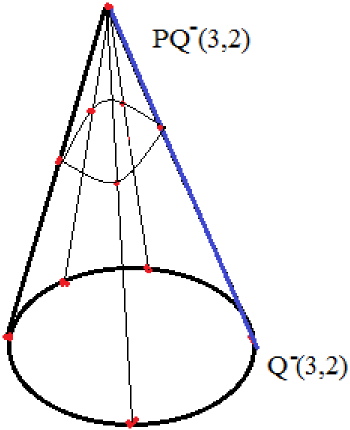
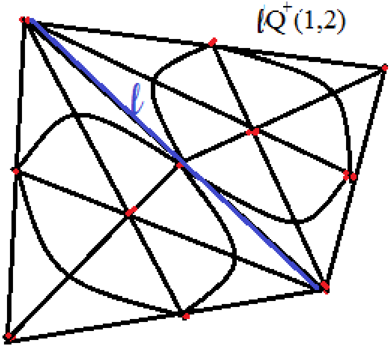
Hence, $\mathcal{G}_3$ *have 10752 cliques of size 5 and 2025 cliques of size 8.*

The isomorphism issue
Mixed type cliques





The isomorphism issue

Lemma

A partial ovoid $\mathcal{O}$ *of* $Q^{+}(7,q)$ *of size* 4 *can be extended in a unique way to an ovoid.*

Lemma

If $\{P,Q,R\}$ *is a 3-clique of mixed type in, then there are exactly two* $\sim_1$*-adjacencies and one* $\sim_2$*-adjacency, and* $\langle P,Q,R\rangle$ *is a plane meeting* Π *in a point* S.

The isomorphism issue

Theorem

The graphs $\mathcal{G}_n$ and $NO^{+}(2n+2,2)$ are not isomorphic when $n \geq 3$.

Remark

$NO^{+}(2n+2,2)$ contains the following number of $\mathcal{G}_3$ subgraphs:

$$(1+o(1))\left(\frac{2^{2n}-1}{2^{2n+1}-2^{n}-1}\right)^{3780}\left(1-\frac{2^{2n}-1}{2^{2n+1}-2^{n}-1}\right)^{3360}\frac{(2^{2n+1}-2^{n})^{120}}{1290240}.$$

$\mathcal{G}_n$ contains the following number of $NO^{+}(8,2)$ subgraphs:

$$(1+o(1))\left(\frac{2^{2n}-1}{2^{2n+1}-2^{n}-1}\right)^{3780}\left(1-\frac{2^{2n}-1}{2^{2n+1}-2^{n}-1}\right)^{3360}\frac{(2^{2n+1}-2^{n})^{120}}{348364800}.$$