Quasi-strongly regular digraphs and new strongly regular digrapha with parameters $(165, 60, 36, 23, 21)$

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5th Pythagorean conference 2025

This work was supported by the Croatian Science Foundation under the project number HRZZ–IP–2022–10–4571.

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Group action

A group G acts on a set S if there exists function $f : G \times S \mapsto S$ such that

1. $f(e, x) = x, \forall x \in S$,
2. $f(g_1, f(g_2, x)) = f(g_1 g_2, x), \forall x \in S, \forall g_1, g_2 \in G$.

Denote the described action by $g.x, \forall x \in S, \forall g \in G$.

The set $G_x = \{g \in G \mid xg = x\}$ is a group called stabilizer of the element $x \in S$. The set $G.x = \{q.x \mid g \in G\}$ is orbit of the element $x \in S$. If there is only one orbit than the action is transitive.

Designs

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is called a t-(v, k, λ) design, if

1. $\mathcal{P}$ contains v points,
2. each block $B \in \mathcal{B}$ is incident with k points, and
3. every t distinct points are incident with λ blocks.

A design is block design if $t = 2$. A design is symmetric if the number of points is equal to the number of blocks.

The incidence matrix of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A bijective mapping of points to points and blocks to blocks which preserves incidence of a design $\mathcal{D}$ is called an automorphism of $\mathcal{D}$. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $\mathrm{Aut}(\mathcal{D})$.

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Construction of 1-designs from groups

Theorem (D. Crnković, VMC, A. Švob)

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \cup_{i=1}^{s} G_\alpha.\delta_i$, where $\delta_1, \dots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g.\Delta_2 \ : \ g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |G_{\delta_i}.\alpha|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \cap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.

D. Crnković, VMC, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3,3)$, J. Statist. Plann. Inference, 144, (2014) 19–40.

D. Crnković, VMC, A. Švob, New 3-designs and 2-designs having $U(3,3)$ as an automorphism group, Discrete Math. 340 (2017), 2507-2515.

D. Crnković, S. Rukavina, A. Švob, New strongly regular graphs from orthogonal groups $O^+(6,2)$ and $O^-(6,2)$, Discrete Math. 341 (2018) 2723-2728.

A. E. Brouwer, D. Crnković, A. Švob, A construction of directed strongly regular graphs with parameters (63,11,8,1,2), Discrete Math. 347 (2024), 114146, 3 pages.

- The incidence matrix of a symmetric 1-design is the adjacency matrix of a regular digraph.
- The incidence matrix of a symmetric 1-design with symmetric incidence matrix is the adjacency matrix of a regular graph.

Quasi-strongly regular digraphs

A quasi-strongly regular digraph[1] (QSRD) $\mathcal{G}$ with parameters $(n, k, t, a; c_1, c_2, \ldots, c_p)$ is a k-regular digraph on n vertices such that

- each vertex is incident with t undirected edges,
- for any two distinct vertices x, y the number of paths of length 2 from x to y is a if $x \rightarrow y$,
- for any two distinct vertices x, y the number of paths of length 2 from x to y is c_i, for $i \in \{1, \ldots, p\}$, if $x \nrightarrow y$
- for each c_i, $i \in \{1, \ldots, p\}$, there exist two distinct vertices $x \nrightarrow y$ such that the number of paths of length 2 from x to y is c_i.

Number p is grade of $\mathcal{G}$ and $c_1 > c_2 > \cdots > c_p$.

[1] D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs, Graphs and Combinatorics*, 38, (2022)

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Number p is grade of $\mathcal{G}$ and $c_1 > c_2 > \cdots > c_p$.

- If $p = 1$, a quasi-strongly regular digraph is strongly regular digraph (SRD) with paramters (n, k, a, c_1, t).
- If $p = 1$ and $k = t$, a quasi-strongly regular digraph is strongly regular graph.

[1] D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs, Graphs and Combinatorics*, 38, (2022)

Construction of k-regular digraphs from groups

Theorem (VMC, Matea Zubović Žutolija)

Let G be a finite permutation group acting transitively on the set Ω. Let $\alpha \in \Omega$ and let $\Delta = \cup_{i=1}^{s} G_\alpha.\delta_i$ be a union of orbits of the stabilizer G_α of α, where $\delta_1, \dots, \delta_s$ are representatives of different G_α-orbits. Let $T = \{g_1, \dots, g_t\}$ be a set of representatives of left cosets in $G/G_\alpha = \{g_1 G_\alpha, \dots, g_t G_\alpha\}$. Let $\mathcal{V} = \{g_i.\alpha | i = 1, \dots, t\}$ and let $\mathcal{E} = \{(g_i.\alpha, g_i.\beta) | i = 1, \dots, t, \beta \in \Delta\}$.
Then $\Gamma = (\mathcal{V}, \mathcal{E})$ is a digraph with $|\Omega|$ vertices that is $|\Delta|$-regular and such that $g_i.\Delta$ is a set of out-neighbours of the vertex $g_i.\alpha$, $i = 1, \dots, t$. The group G acts on the constructed digraph as automorphism group, transitively on the set of vertices.

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Theorem

If a group G acts transitively on a set of vertices of a regular digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then there exists a set Ω such that vertices and arcs of a digraph $\mathcal{G}$ are defined in the way described in Theorem.

New strongly regular digraph

- $G \cong M_{11}$, *i. e.* G is primitive permutation group on 165 points isomorphic to M_{11}
- $\Omega = \{1, ..., 165\}$

- $\Delta = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 20, 21, 22, 23, 30, 36, 37, 38, 45, 57, 58, 59, 66, 72, 73, 74, 75, 76, 77, 84, 85, 86, 87, 94, 95, 96, 97, 98, 99, 105, 111, 118, 119, 121, 122, 123, 130, 142, 143, 144, 145, 146, 147, 148, 155, 156, 162, 164\}$
  - $\Longrightarrow$ Γ is strongly regular digraphs with parameters $(165, 60, 36, 23, 21)$ and a full automorphism group isomorphisc to M_{11}

- $\Delta = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 21, 22, 23, 34, 35, 36, 37, 38, 50, 54, 56, 57, 58, 59, 70, 72, 73, 74, 81, 82, 83, 85, 86, 87, 95, 96, 97, 104, 109, 110, 112, 116, 117, 121, 122, 123, 134, 135, 139, 140, 142, 143, 144, 153, 154, 160, 163, 165\}$
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Classifications

- There are, up to isomorphism, 2920 QSRD on which a transitive automorphism group of degree n, $n \in \{1, \dots, 30\} \setminus \{22, 24, 28, 30\}$, acts. 478 of them are SRD.

- There are, up to isomorphism, 18 QSRD on which the transitive irregular automorphism group of degree 22 acts. Two of them are SRD.
- There are, up to isomorphism, 68235 QSRD on which the transitive irregular automorphism group of degree 24 acts, of which 64 are SRD.
- There are, up to isomorphism, 469 QSRDs on which the transitive irregular automorphism group of degree 28 acts, of which 22 are SRD.
- There are, up to isomorphism, 642 QSRD on which the transitive irregular automorphism group of degree 30 acts, of which 12 are SRD.

- There are, up to isomorphism, 124 QSRD, on which the primitive automorphism group of degree n, $n \in \{31, \dots, 110\}$ acts, no SRD.

More results

- There is no SRD with parameters $(22, 9, 3, 4, 6)$ [2] such that the automorphism group G acts transitively on the set of vertices of that digraph.
- There is no SRD with parameters $(24, 10, 3, 5, 5)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.
- There is no SRD with parameters $(28, 6, 2, 1, 3)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.
- There is no SRD with parameters $(30, 11, 2, 5, 9)$ and $(30, 12, 4, 5, 11)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.

[2] V. A. Byzov, I. A. Pushkarev, *On the existence of directed strongly regular graphs with parameters* (22,9,6,3,4), 2024

Thank you for your attention!