

Quasi-strongly regular digraphs and new strongly regular digraphs with parameters $(165, 60, 36, 23, 21)$

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Groups and designs

Construction of 1-designs from groups

Digraphs

Construction of k -regular digraphs from groups

Constructed directed regular graphs

Group action

A group G **acts** on a set S if there exists function $f : G \times S \mapsto S$ such that

1. $f(e, x) = x, \forall x \in S$,
2. $f(g_1, f(g_2, x)) = f(g_1 g_2, x), \forall x \in S, \forall g_1, g_2 \in G$.

Denote the described action by $g.x, \forall x \in S, \forall g \in G$.

The set $G_x = \{g \in G \mid xg = x\}$ is a group called **stabilizer** of the element $x \in S$.

The set $G.x = \{g.x \mid g \in G\}$ is **orbit** of the element $x \in S$. If there is only one orbit than the action is **transitive**.



Designs

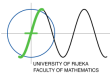
An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set \mathcal{P} , block set \mathcal{B} and incidence \mathcal{I} is called a t -(v, k, λ) design, if

1. \mathcal{P} contains v points,
2. each block $B \in \mathcal{B}$ is incident with k points, and
3. every t distinct points are incident with λ blocks.

A design is **block design** if $t = 2$. A design is **symmetric** if the number of points is equal to the number of blocks.

The **incidence matrix** of a design is a $b \times v$ matrix $[m_{ij}]$ where b and v are the numbers of blocks and points respectively, such that $m_{ij} = 1$ if the point P_j and the block B_i are incident, and $m_{ij} = 0$ otherwise.

A bijective mapping of points to points and blocks to blocks which preserves incidence of a design \mathcal{D} is called an **automorphism** of \mathcal{D} . The set of all automorphisms of \mathcal{D} forms its full automorphism group denoted by $\text{Aut}(\mathcal{D})$.



Construction of 1-designs from groups

Theorem (D. Crnković, VMC, A. Švob)

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n , respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \cup_{i=1}^s G_{\alpha} \cdot \delta_i$, where $\delta_1, \dots, \delta_s \in \Omega_2$ are representatives of distinct G_{α} -orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g \cdot \Delta_2 : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_{\alpha}|}{|G_{\Delta_2}|} \sum_{i=1}^s |G_{\delta_i} \cdot \alpha|)$ design with $\frac{m \cdot |G_{\alpha}|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \cap_{x \in \Omega_2} G_x$ acts as an automorphism group on (Ω_2, \mathcal{B}) , transitively on points and blocks of the design.



D. Crnković, VMC, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3, 3)$, J. Statist. Plann. Inference, 144, (2014) 19–40.



D. Crnković, VMC, A. Švob, New 3-designs and 2-designs having $U(3, 3)$ as an automorphism group, Discrete Math. 340 (2017), 2507-2515.



D. Crnković, S. Rukavina, A. Švob, New strongly regular graphs from orthogonal groups $O^+(6, 2)$ and $O^-(6, 2)$, Discrete Math. 341 (2018) 2723-2728.



A. E. Brouwer, D. Crnković, A. Švob, A construction of directed strongly regular graphs with parameters (63,11,8,1,2), Discrete Math. 347 (2024), 114146, 3 pages.

- ▶ The incidence matrix of a symmetric 1-design is the adjacency matrix of a regular digraph.
- ▶ The incidence matrix of a symmetric 1-design with symmetric incidence matrix is the adjacency matrix of a regular graph.

Quasi-strongly regular digraphs

A **quasi-strongly regular digraph**¹ (QSRD) \mathcal{G} with parameters $(n, k, t, a; c_1, c_2, \dots, c_p)$ is a k -regular digraph on n vertices such that

- ▶ each vertex is incident with t undirected edges,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is a if $x \rightarrow y$,
- ▶ for any two distinct vertices x, y the number of paths of length 2 from x to y is c_i , for $i \in \{1, \dots, p\}$, if $x \nrightarrow y$
- ▶ for each c_i , $i \in \{1, \dots, p\}$, there exist two distinct vertices $x \nrightarrow y$ such that the number of paths of length 2 from x to y is c_i .

Number p is **grade** of \mathcal{G} and $c_1 > c_2 > \dots > c_p$.

¹D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs*, *Graphs and Combinatorics*, **38** (2022)

Quasi-strongly regular digraphs

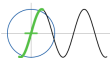
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Number p is **grade** of \mathcal{G} and $c_1 > c_2 > \dots > c_p$.

- ▶ If $p = 1$, a quasi-strongly regular digraph is **strongly regular digraph** (SRD) with parameters (n, k, a, c_1, t) .
- ▶ If $p = 1$ and $k = t$, a quasi-strongly regular digraph is **strongly regular graph**.

¹D. Jia, Z. Guo, G. Zhang, *Some constructions of quasi-strongly regular graphs*, *Graphs and Combinatorics*, **38** (2022)



Construction of k -regular digraphs from groups

Theorem (VMC, Matea Zubović Žutolija)

Let G be a finite permutation group acting transitively on the set Ω . Let $\alpha \in \Omega$ and let $\Delta = \cup_{i=1}^s G_\alpha \cdot \delta_i$ be a union of orbits of the stabilizer G_α of α , where $\delta_1, \dots, \delta_s$ are representatives of different G_α -orbits. Let $T = \{g_1, \dots, g_t\}$ be a set of representatives of left cosets in $G/G_\alpha = \{g_1 G_\alpha, \dots, g_t G_\alpha\}$. Let $\mathcal{V} = \{g_i \cdot \alpha \mid i = 1, \dots, t\}$ and let $\mathcal{E} = \{(g_i \cdot \alpha, g_i \cdot \beta) \mid i = 1, \dots, t, \beta \in \Delta\}$.

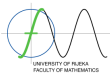
Then $\Gamma = (\mathcal{V}, \mathcal{E})$ is a digraph with $|\Omega|$ vertices that is $|\Delta|$ -regular and such that $g_i \cdot \Delta$ is a set of out-neighbours of the vertex $g_i \cdot \alpha$, $i = 1, \dots, t$. The group G acts on the constructed digraph as automorphism group, transitively on the set of vertices.

Theorem

If a group G acts transitively on a set of vertices of a regular digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then there exists a set Ω such that vertices and arcs of a digraph \mathcal{G} are defined in the way described in Theorem.

New strongly regular digraph

- ▶ $G \cong M_{11}$, i. e. G is primitive permutation group on 165 points isomorphic to M_{11}
- ▶ $\Omega = \{1, \dots, 165\}$
- ▶ $\Delta = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 20, 21, 22, 23, 30, 36, 37, 38, 45, 57, 58, 59, 66, 72, 73, 74, 75, 76, 77, 84, 85, 86, 87, 94, 95, 96, 97, 98, 99, 105, 111, 118, 119, 121, 122, 123, 130, 142, 143, 144, 145, 146, 147, 148, 155, 156, 162, 164\}$
 - ▶ $\implies \Gamma$ is strongly regular digraphs with parameters (165, 60, 36, 23, 21) and a full automorphism group isomorphic to M_{11}
- ▶ $\Delta = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 21, 22, 23, 34, 35, 36, 37, 38, 50, 54, 56, 57, 58, 59, 70, 72, 73, 74, 81, 82, 83, 85, 86, 87, 95, 96, 97, 104, 109, 110, 112, 116, 117, 121, 122, 123, 134, 135, 139, 140, 142, 143, 144, 153, 154, 160, 163, 165\}$
 - ▶ $\implies \Gamma$ is strongly regular digraphs with parameters (165, 60, 36, 23, 21) and a full automorphism group isomorphic to M_{11}



Classifications

- ▶ There are, up to isomorphism, 2920 QSRD on which a transitive automorphism group of degree n , $n \in \{1, \dots, 30\} \setminus \{22, 24, 28, 30\}$, acts. 478 of them are SRD.
- ▶ There are, up to isomorphism, 18 QSRD on which the transitive irregular automorphism group of degree 22 acts. Two of them are SRD.
- ▶ There are, up to isomorphism, 68235 QSRD on which the transitive irregular automorphism group of degree 24 acts, of which 64 are SRD.
- ▶ There are, up to isomorphism, 469 QSRDs on which the transitive irregular automorphism group of degree 28 acts, of which 22 are SRD.
- ▶ There are, up to isomorphism, 642 QSRD on which the transitive irregular automorphism group of degree 30 acts, of which 12 are SRD.
- ▶ There are, up to isomorphism, 124 QSRD, on which the primitive automorphism group of degree n , $n \in \{31, \dots, 110\}$ acts, no SRD.

More results

- ▶ There is no SRD with parameters $(22, 9, 3, 4, 6)$ ² such that the automorphism group G acts transitively on the set of vertices of that digraph.
- ▶ There is no SRD with parameters $(24, 10, 3, 5, 5)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.
- ▶ There is no SRD with parameters $(28, 6, 2, 1, 3)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.
- ▶ There is no SRD with parameters $(30, 11, 2, 5, 9)$ and $(30, 12, 4, 5, 11)$ such that the automorphism group G acts transitively on the set of vertices of that digraph.

²V. A. Byzov, I. A. Pushkarev, *On the existence of directed strongly regular graphs with parameters $(22, 9, 6, 3, 4)$* , 2024

Thank you for your attention!