On the chromatic number of Grassmann graphs

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(joint work with Jozefien D'haeseleer)

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Overview

Basic Graph Theory Notions

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A **finite simple graph** $\Gamma = \Gamma(V, E)$ is a pair, where V is a finite set of vertices, and $E \subset \binom{V}{2}$ is the set of edges.

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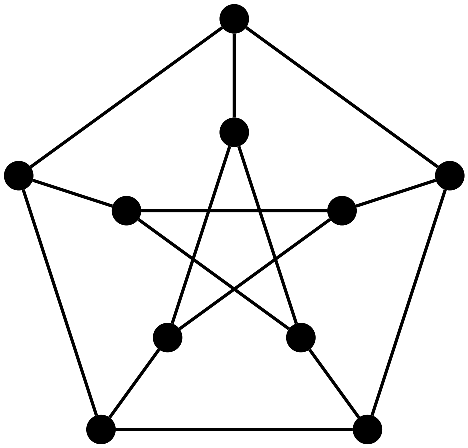
Definition.
The **chromatic number** of a graph G, denoted $\chi(G)$, is the smallest positive integer k for which there exists a valid coloring of G using k colors.

Example: The Petersen Graph



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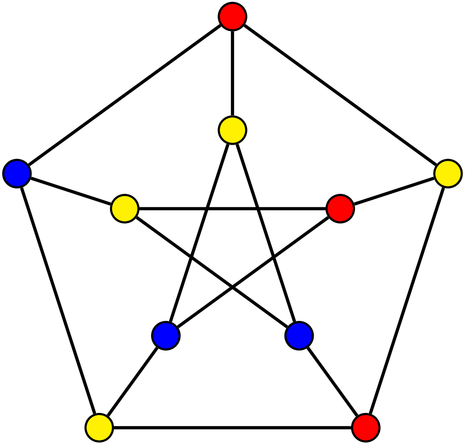

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- Let $\omega(G)$ be the clique number of G. Then $\omega(G) \leq \chi(G)$.
- Let $\alpha(G)$ be the independence number of G. Then

$$\chi(G)\alpha(G) \geq n.$$

Johnson Graphs

Definition.
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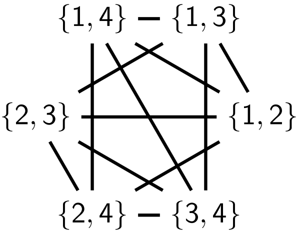

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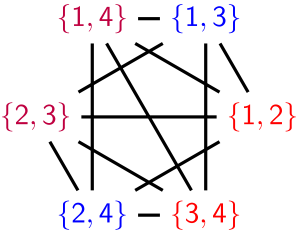

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Upper Bound: Color classes are elements of $\mathbb{Z}_n$. Assign the color $a_1 + a_2 + \cdots + a_m \pmod{n}$ to the subset $\{a_1, a_2, \ldots, a_m\}$.

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A few more results are known depending on the relationship between n and m, see Etzion 1992, Etzion and Bitan 1996.

The Grassmann Graph

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Let q be a prime power and n a positive integer. The **Grassmann graph** $J_q(n, m)$ has as vertices the collection of all m-dimensional subspaces of $\mathbb{F}_q^n$ and an edge is placed between two vertices when the corresponding subspaces intersect in an $(m-1)$-dimensional subspace. Denote this graph by $J_q(n, m)$

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- When $m=2$, $J_q(n,2)$ is the line incidence graph of the projective space $PG(n-1,q)$.

The **Gaussian binomial** coefficient is defined as

$$\binom{n}{m}_q := \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-m+1}-1)}{(q-1)(q^2-1)\cdots(q^m-1)}$$

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- The clique number of $J_q(n,m)$ is at least $\binom{n-m+1}{1}_q$.

A General Bound

Theorem (D'haeseleer and T. (2025+)).
Let q be a prime power and $m < n$ be positive integers, then

$$\binom{n-m+1}{1}_q \leq \chi(J_q(n,m)) \leq \binom{n}{1}_q .$$

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$$\begin{vmatrix} x_1 & x_2 & \cdots & x_m \\ x_1^q & x_2^q & \cdots & x_m^q \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{q^{m-1}} & x_2^{q^{m-1}} & \cdots & x_m^{q^{m-1}} \end{vmatrix} \in C.$$

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- This is can equivalently be thought of as a partition of the points of $\mathrm{PG}(n-1,q)$ into skew parallel lines.
- A simple counting argument implies spreads can only exist in $\mathbb{F}_q^n$ if n is even.
- On the other hand, when n is even, identifying $\mathbb{F}_q^n$ with $\mathbb{F}_{q^n}$ and adding the element 0 to each of the cosets $\mathbb{F}_{q^n}^*/\mathbb{F}_{q^2}^*$ yields a spread, implying spreads exist when n is even.

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- Note that a spread of $\mathrm{PG}(n-1,q)$ is a maximal independent set in $J_q(n,2)$.
- Consequently, there exists parallelism in $PG(n-1,q)$ if and only if $\chi(J_q(n,2)) = \binom{n-1}{1}_q$.

Results on Line Parallelisms

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3. When $q = 3, 4, 8, 16$ and n is any positive integer. (Feng, Xu 2023)

Furthermore, when $q = 2$ and n is odd, no parallelisms exist but it has been determined that $\chi(J_2(n, 2)) = \binom{n-1}{1}_2 + 3$ (Meszka 2013).

Theorem (D'haeseleer and T. (2025+)).
Let e be any positive integer, n be an even integer and $q = 2^e$. Then

$$\chi(J_q(n,2)) < 2\binom{n-1}{1}_q$$

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Proof Idea: Demonstrate a homomorphism from $J_q(n,2)$ into the 2-Kneser graph $K_2((n-1)e, e)$. Therefore we have

$$\chi(J_q(n,m)) \leq \chi(K_2((n-1)e, e)) = \binom{(n-2)e+1}{1} < 2\binom{n-1}{1}_q.$$

Conclusion and Further Research

- Small computations suggest that chromatic number of the subgraph of the Kneser graph induced by this homomorphism is strictly greater than $\binom{n-1}{1}_q$, so a different function (or entirely different method) needs to be used if one wishes to prove that the lower bound is the true answer.

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- Can we give new constructions of parallelisms in $\mathbb{F}_2^n$?

Questions?

Thank you!