On the Existence of Lattice Tiling of Lee Spheres

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Outline

- Introduction
- From Lattice Tiling to Polynomials
- Necessary Conditions
- Computational Results

1. Introduction

Introduction

- For any $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{Z}^n$, the **Lee distance** ($\ell_1$-norm, Manhatten distance…) between them is $d_L(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.
- Lee sphere of radius r centered at O is:

$$\boldsymbol{B(n,r)} := \{(x_1, \cdots, x_n) \in \mathbb{Z}^n\colon\ \sum_{i=1}^{n} |x_i| \ \le r\}$$

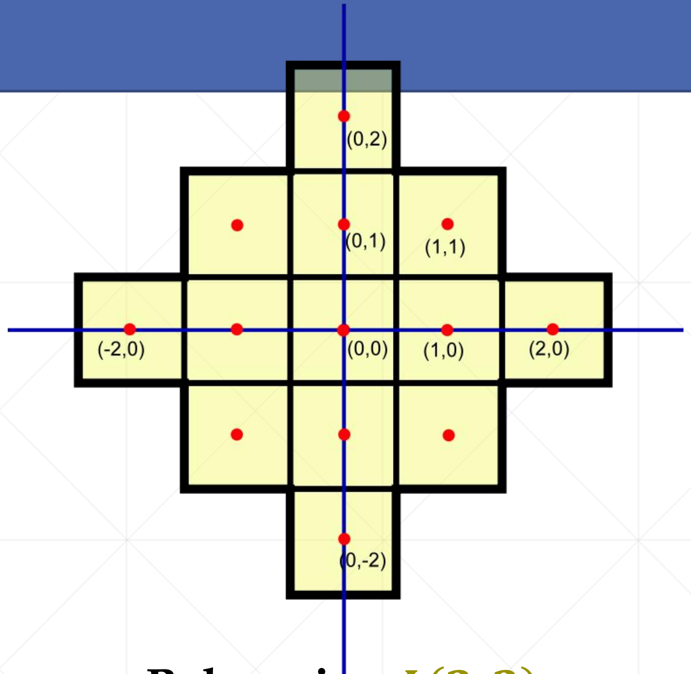
$$|B(n,r)| = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{n}{i}\binom{r}{i}.$$

- A **perfect Lee code** C ⇔ A **tiling** of $\mathbb{Z}^n$ by translates of $B(n,r)$

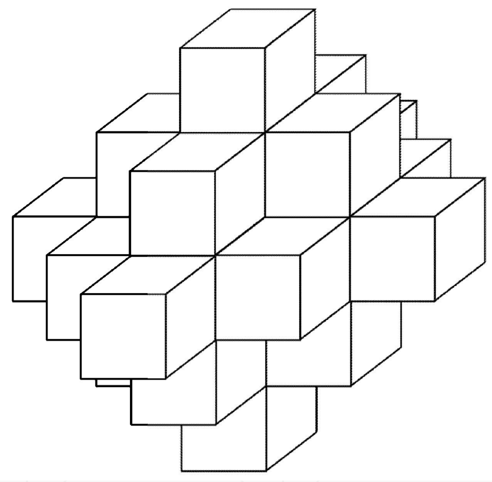
$$\mathbb{Z}^n = \dot{\bigcup}_{c\in C}(B(n,r) + c) = B(n,r) \oplus C$$

- It is equivalent to ``tile" $\mathbb{R}^n$ by $\boldsymbol{L(n,r)} = \boldsymbol{B(n,r)} + \left[-\frac{1}{2}, \frac{1}{2}\right]^n$

$$\mathbb{R}^n =_{a.e.} \mathbb{Z}^n + \left[-\frac{1}{2}, \frac{1}{2}\right]^n =_{a.e.} \boldsymbol{L(n,r)} \oplus C$$



Polyomino $L(2,2)$ associated with $B(2,2)$

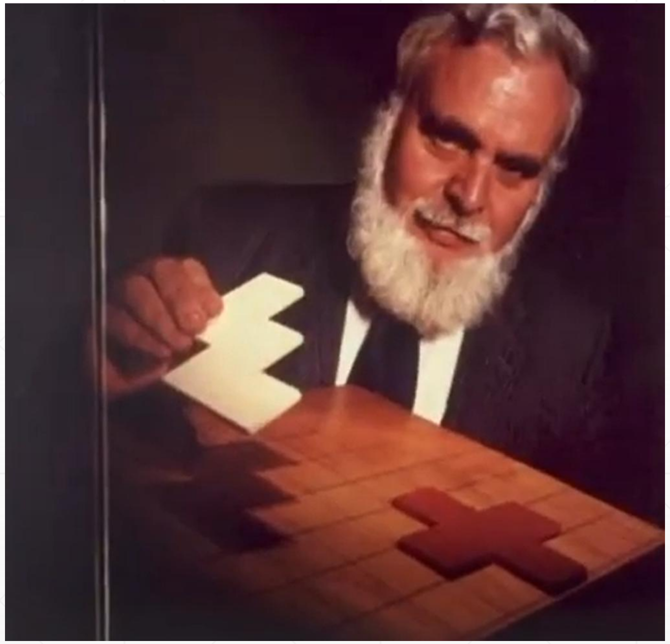


Polyomino $L(3,2)$

Introduction

- **Theorem (Golomb, Welch 1968/1970)** Perfect Lee codes exist for $n = 1,2$ and any r; and for $r = 1$ and any n.

- **Golomb-Welch conjecture**: there are no more perfect Lee codes for other choices of n and r.



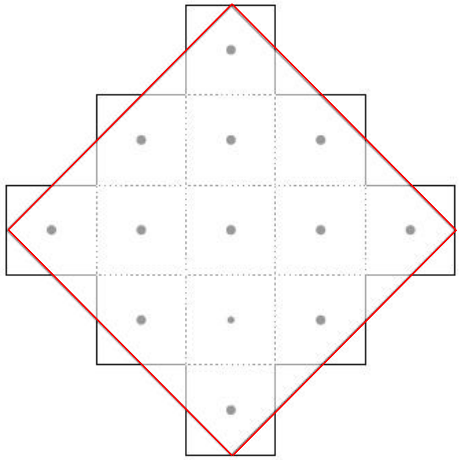
Solomon Golomb (1932-2016)

5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4
8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7
11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2
6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5
9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8
12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3
7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6
10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1

$n = 2$, $r = 2$, the green points form a perfect code (lattice)

Introduction

- **GW conjecture**: ∄ Perfect Lee codes for $n \geq 3$ and $r \geq 2$.
- partially proved for given n and $r > N(n)$.

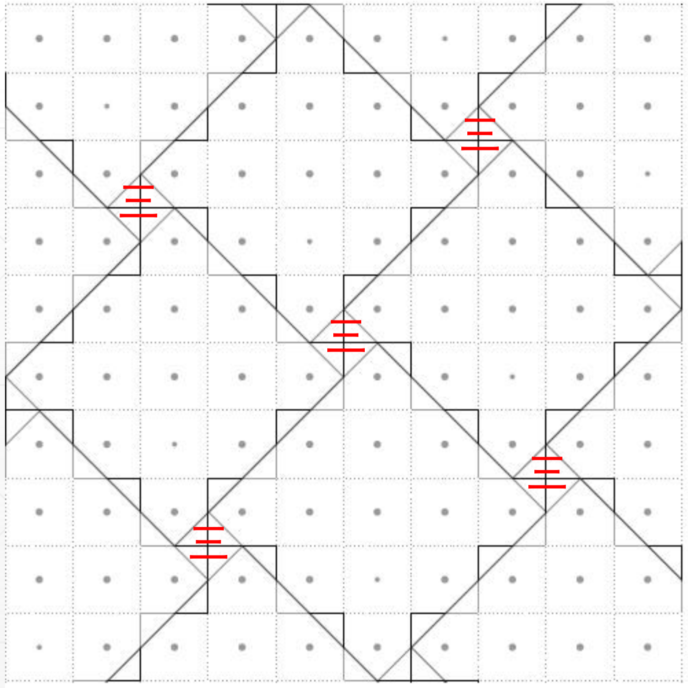


Basic Idea by GW

- Cross-polytope: convex hull of $\{\pm(0,\cdots,\overset{i}{1},\cdots,0): i = 1,\cdots n\}$.
- Conscribed cross-polytope $X(n,r)$ of $L(n,r)$,

$$\mathrm{vol}\big(X(n,r)\big) = \frac{(2r+1)^n}{n!}$$

$$\mathrm{vol}\big(L(n,r)\big) = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{r}{i}\binom{n}{i} \approx 2^n \binom{r}{n}, r \to \infty.$$



- The <u>packing density</u> of $X(n,r)$ must be smaller than $(0.87)^n$, n large enough. (Tóth, Fodor, Vígh, 2015)

Introduction

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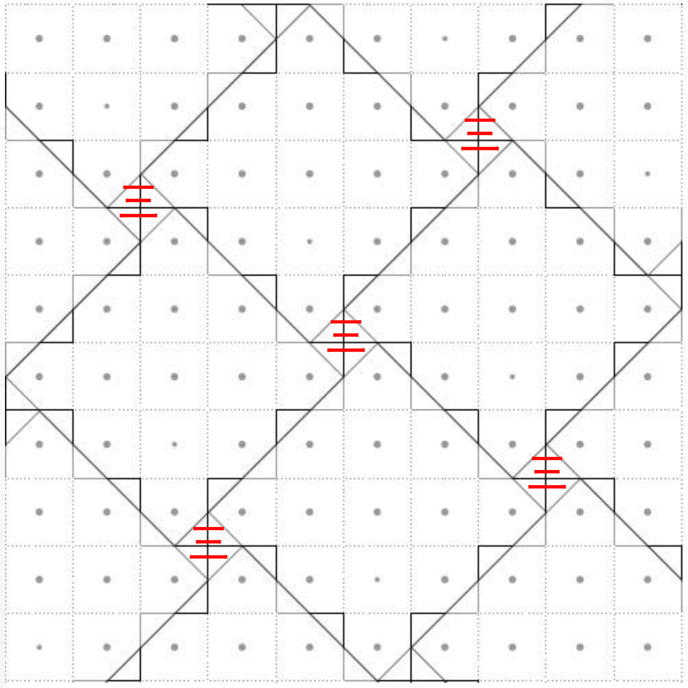
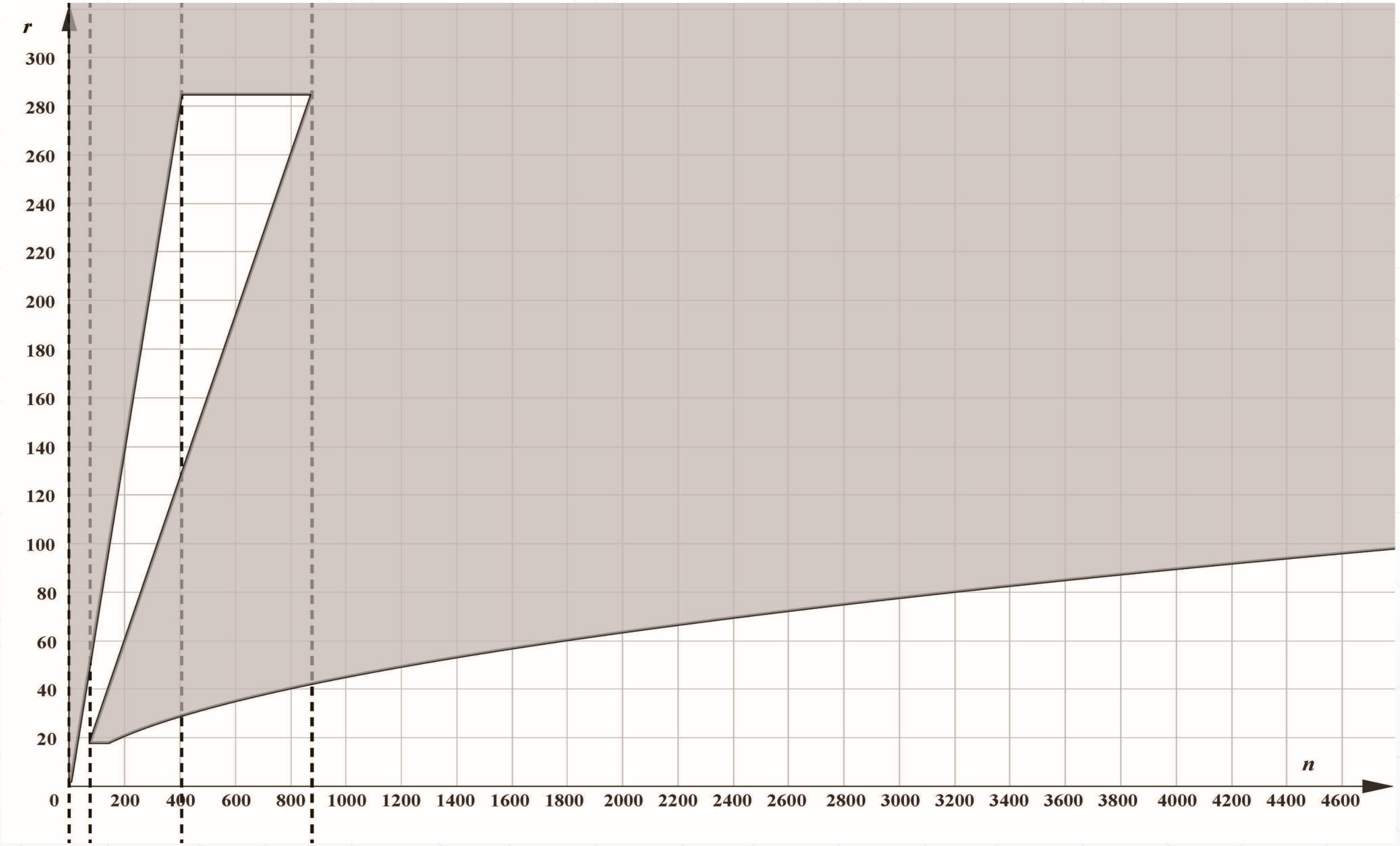


- The packing density of $X(n,r)$ must be smaller than $(0.87)^n$, n large enough. (Tóth, Fodor, Vígh, 2015)

Introduction

GW conjecture was partially proved for given n and $r > N(n)$:



- $3 \le n \le 74$, $\max\left\{\frac{\sqrt{2}}{2}n - \frac{3}{4}\sqrt{2} - \frac{1}{2}, 2\right\} \le r$.
- $75 \le n \le 405$, $\max\{18, \sqrt{2n+40}\} \le r \le \frac{n-21}{3}$

  or $\frac{\sqrt{2}}{2}n - \frac{3}{4}\sqrt{2} - \frac{1}{2} \le r$.
- $406 \le n \le 876$, $\sqrt{2n+40} \le r \le \frac{n-21}{3}$

  or $285 \le r$.
- $n \ge 876$, $\sqrt{2n+40} \le r$.

Reference: Horak, Kim. 50 years of the Golomb-Welch conjecture. IEEE TIT 64(2), 2018 and references therein

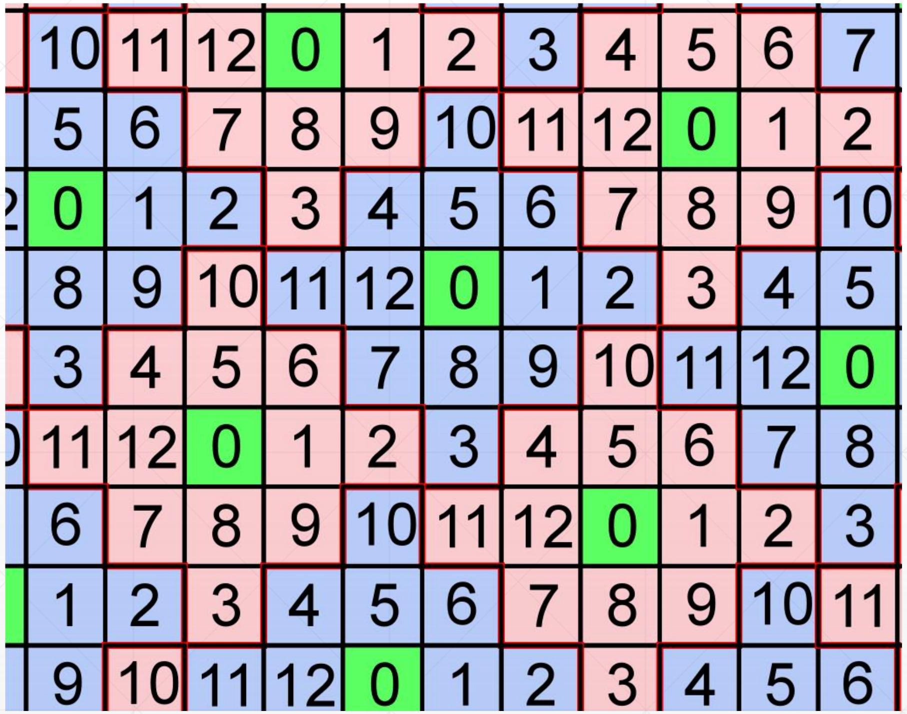
2. From Lattice Tiling to Polynomials

Introduction

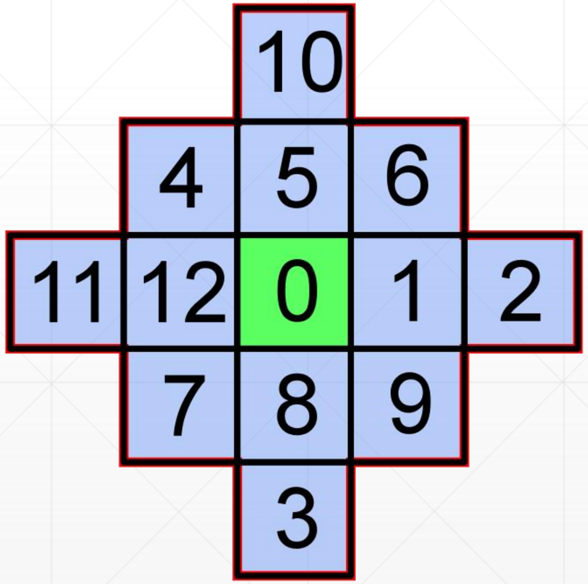
- A **lattice tiling** of $\mathbb{Z}^n$ by translates of $B(n,r)$

$$\mathbb{Z}^n = \dot{\bigcup}_{c \in C}(B(n,r) + c) = B(n,r) \oplus C, \text{ and } C \subseteq \mathbb{Z}^n \text{ is a lattice}.$$

Theorem 1 (Horak, AlBdaiwi 2012) $\exists$ a lattice tiling of $\mathbb{Z}^n$ by Lee spheres of radius r $\Leftrightarrow$ there are an abelian group G of order $|B(n,r)|$ and a homomorphism $\varphi: \mathbb{Z}^n \mapsto G$ such that $\varphi|_{B(n,r)}$ is a bijection.

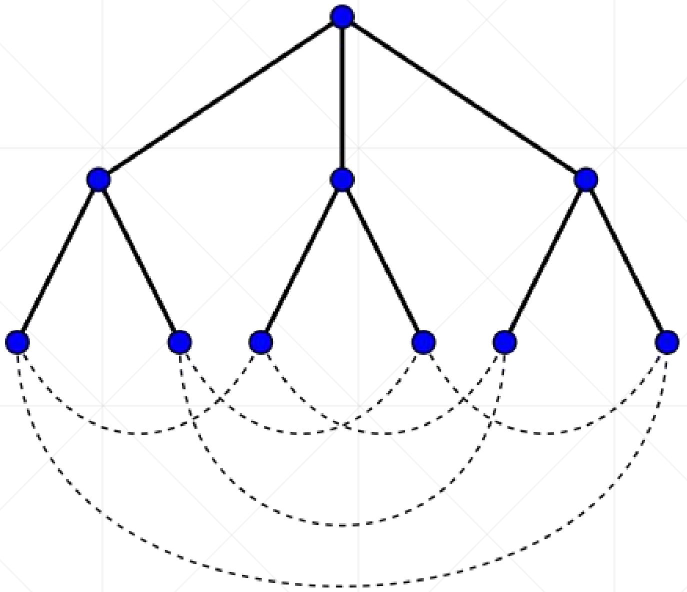


$n = 2,\ r = 2$



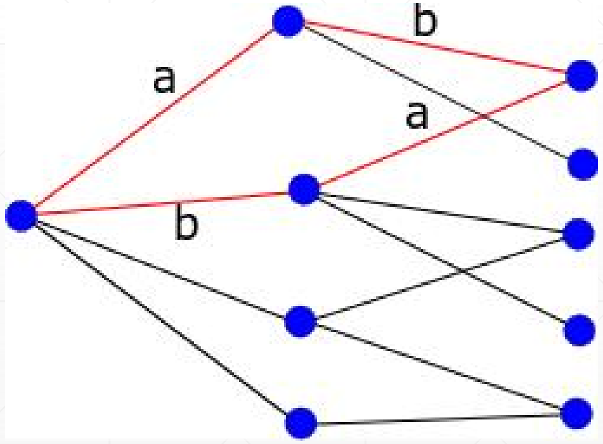
$G = (\mathbb{Z}_{13},+),\ \varphi(e_1) = 1,\ \varphi(e_2) = 5$

A related problem in Graph Theory



Degree/diameter problems

- **Moore bound** for general graphs: $\#V \leq 1 + d\sum_{i=0}^{k-1}(d-1)^i$.
- **Moore-like bound** for abelian Cayley graphs:



$$|G| \leq \sum_{i=0}^{\min\{k,d\}} 2^i \binom{k}{i}\binom{d}{i}.$$

- Moore-like bound = $|B(n,r)|$ with $n = d, r = k$.
- Abelian Cayley graph meeting the Moore-like bound⇔Lattice tiling of $B(n,r)$

Lattice Tiling to Polynomials

Recall: **geometric method** can only handle GW-conjecture for fixed n and $r > N(n)$.

For lattice GW-conjecture with fixed r, **Algebraic and Combinatorics Methods:**

- Symmetric polynomials over finite fields (Kim 2017, Zhang, Ge 2017, Qureshi 2020)
  - Fast algorithm for small n;
  - Works for infinitely many n? Some times.
  - Usually, $|B(n,r)|$ needs to be prime or to have large prime divisors.
- Convert the original problem into a group ring equation
  - Group characters (=eigenvalue of the associated graph), algebraic number theory, finite fields...(Zhang, Z. 2019)
    - Usually need small prime divisors of $|B(n,r)|$.
  - Handle the group ring equations directly mod 3, mod 5... (Leung, Z. 2020)
    - Currently only works for $r = 2$ and all $n \geq 3$.

Group Ring Equations Approach

A lattice tiling of Lee spheres of radius 2 in $\mathbb{Z}^n$ ⇔ The existence of $T \subseteq G$, where G is an abelian (**multiplicative**) group of order $2n^2 + 2n + 1$, such that $T = T^{(-1)}$, the identity $e \in T$ and

$$T^2 = 2G - T^{(2)} + 2ne \in \mathbb{Z}[G],$$

where $T^{(s)} := \sum_{t \in T} t^s$.

[Zhang & Z. 2019] Apply $\chi \in \widehat{G}$, obtain equations in algebraic integer rings.

[Leung & Z. 2020] Analyze $T^3 \equiv T^{(3)} \bmod 3, T^5 \equiv T^{(5)} \bmod 5$

- $r = 3$: $T^3 = 6G - 3T^{(2)}T - 2T^{(3)} + 6nT$;
- $r = 4$:
  $T^4 = 24G - 12n\left(T^2 + T^{(2)}\right) - 6T^{(2)}T^2 - 3T^{(2)}T^{(2)} - 8\,T^{(3)}T - 6T^{(4)} + 12n(n-1)$;
- $r = 5$: ……

Symmetric polynomial approach

Theorem 1 The following three conditions are equivalent:

a) $\exists$ a lattice tiling of $\mathbb{Z}^n$ by Lee spheres of radius r

b) there are an abelian group G of order $|B(n,r)|$ and a homomorphism $\varphi : \mathbb{Z}^n \mapsto G$ such that $\varphi|_{B(n,r)}$ is a bijection

c) $\exists$ abelian (**additive**) group G of order $|B(n,r)|$ and $\exists R = \{x_1, \ldots, x_n\} \subseteq G$, such that $\left\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^n, \|u\|_1 \leq r\right\} = G$.

10	11	12	0	1	2	3	4	5	6	7
5	6	7	8	9	10	11	12	0	1	2
0	1	2	3	4	5	6	7	8	9	10
8	9	10	11	12	0	1	2	3	4	5
3	4	5	6	7	8	9	10	11	12	0
11	12	0	1	2	3	4	5	6	7	8
6	7	8	9	10	11	12	0	1	2	3
1	2	3	4	5	6	7	8	9	10	11
9	10	11	12	0	1	2	3	4	5	6

Example: $R = \{1,5\} \subseteq G = C_{13}$.

$$\left\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^2, \|u\|_1 \leq 2\right\} = \{0, \pm 1, \pm 5, \pm 2, \pm 10, \pm 1 \pm 5\} = C_{13}$$

Lattice Tiling to Polynomials

$R = \{x_1, ..., x_n\} \subseteq G$, $\left\{\sum_{x_i \in R} u_i x_i : u \in \mathbb{Z}^n, \|u\|_1 \le r\right\} = G$ with $|G| = |B(n,r)| = \sum_{i=0}^{\min(n,r)} 2^i \binom{n}{i}\binom{r}{i}$.

The idea by Kim ($r = 2$), generalized by Zhang & Ge, and Qureshi ($r \ge 2$):

- Suppose that $|G| = pm$, define projection $\varphi: G \to (\mathbb{F}_p, +)$, $\bar{x} \coloneqq \varphi(x)$. Consider

$$Q_{(n,r)}^k(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{u \in \mathbb{Z}^n : \|u\|_1 \le r} \left(\varphi\left(\sum_{x_i \in R} u_i x_i\right)\right)^{2k} = \sum_{u \in \mathbb{Z}^n : \|u\|_1 \le r} \left(\sum_{x_i \in R} u_i \bar{x}_i\right)^{2k}$$

$$= \sum_{g \in G} \varphi(g)^{2k} = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

$$\sum_{x \in \mathbb{F}_p^*} x^j = \begin{cases} 0, & p-1 \nmid j; \\ -1, & p-1 \mid j. \end{cases}$$

- By expanding,

$$Q_{(n,r)}^k(\bar{x}_1, \cdots, \bar{x}_n) = \sum_\lambda c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \ne (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n),$$

where $S_\lambda = S_{\lambda_1} \cdots S_{\lambda_\ell}$ and $(\lambda_1, \cdots, \lambda_\ell)$ is a partition of $2k$ with $\ell \le r$ and $S_m(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{i=1}^n \bar{x}_i^m$.

Examples

For $r = 2$,

$$Q_{(n,2)}^{k}(X) = (4^k + 4n + 2)S_{2k} + 2\sum_{t=1}^{k-1}\binom{2k}{2t}S_{2t}S_{2(k-t)} = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

For $r = 3$,

$$\begin{aligned} Q_{(n,3)}^{k}(X) \\ &= \left(\frac{2\times 9^k}{3} + (2n+1)4^k + 4n^2 + 4n + 2\right)S_{2k} \\ &+ \sum_{t=1}^{k-1}(4^t + 4^{k-t} + 4n + 2)\frac{(2k)!}{(2t)!\,(2k-2t)!}S_{2t}S_{2k-2t} \\ &+ \frac{4}{3}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\frac{(2k)!}{(2j)!\,(2i-2j)!\,(2k-2i)!}S_{2j}S_{2i-2j}S_{2k-2i} = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases} \end{aligned}$$

Lattice Tiling to Polynomials

A **necessary condition** for the existence of lattice tiling of $B(n,r)$:

$|G| = |B(n,r)| = pm$, projection $\varphi: G \to (\mathbb{F}_p, +)$, $\bar{x} := \varphi(x)$. Consider $Q^k_{(n,r)}(\bar{x}_1, \cdots, \bar{x}_n)$.

For $r = 2$, $Q^k_{(n,2)}(\bar{x}_1, \cdots, \bar{x}_n) = (4^k + 4n + 2)S_{2k} + 2\sum_{t=1}^{k-1} \binom{2k}{2t} S_{2t}S_{2(k-t)} = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$

Key idea: if $4^k + 4n + 2 \not\equiv 0 \pmod{p}$ for $k = 1,2, \ldots$, then recursively we get

$$k = 1: \quad (4^1 + 4n + 2)S_{2\cdot 1} = 0 \Rightarrow S_2 = 0$$

$$k = 2: \quad (4^2 + 4n + 2)S_{2\cdot 2} + 2\binom{4}{2} S_2 S_2 = 0 \Rightarrow S_{2\cdot 2} = 0$$

$$\vdots$$

- [Kim 2017] For $m = 1$, i.e., $|G| = p$ and the assumption holds for $k = 1,2, \ldots n-1$, we obtain

$$S_2, S_4, \ldots, S_{2n} = 0 .$$

Then $e_n = x_1^2 \cdots x_n^2 = 0$ (by Newton's identity), where e_i is the elementary symmetric polynomial of $x_1^2, \cdots, x_n^2$, then a contradiction!

Lattice Tiling to Polynomials

A **necessary condition** for the existence of lattice tiling of $B(n,r)$:

$|G| = |B(n,r)| = pm$, projection $\varphi: G \to (\mathbb{F}_p, +)$, $\bar{x} := \varphi(x)$.

$$Q^k_{(n,r)}(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{\lambda} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n)$$

$$= c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \neq (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

where $S_\lambda = S_{\lambda_1} \cdots S_{\lambda_\ell}$ and $(\lambda_1, \cdots, \lambda_\ell)$ is a partition of $2k$ with $\ell \leq r$ and $S_m(\bar{x}_1, \cdots, \bar{x}_n) = \sum_{i=1}^n \bar{x}_i^m$.

[Qureshi 2020]:

a) Determines the exact value of $c_{(2k)}$.

b) If $p \nmid m$ and the leading coefficients $c_{(2k)} \neq 0$ in $\mathbb{F}_p$ for $k = 1, \cdots, \frac{p-1}{2}$ and $c_{(p-1)} = 0$, then $S_2 = S_4 = \cdots = S_{p-3} = 0$ which implies $c_{(p-1)} S_{p-1} =- m \neq 0$, a contradiction!

Lattice Tiling to Polynomials

$$Q^k_{(n,r)}(\bar{x}_1,\cdots,\bar{x}_n) = c_{(2k)}S_{2k} + \sum_{\lambda\neq(2k)} c_\lambda S_\lambda = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

- [Kim 2017] $r = 2$, $c_{(2k)} \not\equiv 0 \pmod p$ for $k = 1,2,\ldots,n-1$.
- [Qureshi 2020] $\forall\, r$, $c_{(2k)} \not\equiv 0 \pmod p$ for $k = 1,\cdots,\frac{p-1}{2} - 1$ and $c_{(p-1)} \equiv 0 \not\equiv -m$.

Main problems: In the general case with $c_\lambda \equiv 0 \pmod p$ for some λ, can we still derive contradictions?

Two main tasks:

a) Determine the recursive formula for S_λ, i.e., the exact value of every c_λ;

b) Use $[S_2, S_4, S_6, \cdots]$ to derive contradictions about $\bar{x}_1,\cdots,\bar{x}_n$

3. Necessary Conditions

Discrete Fourier Analysis

From $S_{2k} = \sum \bar{x}_i^{2k}$, what we can get?

- [Kim 2017]. Using Newton's identity, we can get the value of elementary symmetric polynomials on $\bar{x}_i$'s partially ($k \neq 0 \bmod p$).

$$ke_k(\bar{x}_1^2, \bar{x}_2^2, \dots, \bar{x}_n^2) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(\bar{x}_1^2, \bar{x}_2^2, \dots, \bar{x}_n^2) S_i(\bar{x}_1^2, \bar{x}_2^2, \dots, \bar{x}_n^2)$$

In fact, we can determine $\bar{x}_i^2$ **completely**! Set $\boxtimes_p = \{z^2 : z \in \mathbb{F}_p^*\}$, and

$$\chi_k : \boxtimes_p \to \mathbb{F}_p$$
$$z \mapsto z^k$$

- It is a character from group $\boxtimes_p$ (under multiplication) of order $\frac{p-1}{2}$ to $\mathbb{F}_p$.
- Character Group: $\widehat{\boxtimes}_p = \left\{\chi_k : k = 0, \dots, \frac{p-1}{2} - 1\right\}$.

Discrete Fourier Analysis

$|G| = |B(n,r)| = pm$, projection $\varphi: G \to (\mathbb{F}_p, +), \bar{x} := \varphi(x)$, $R = \{x_1, \ldots, x_n\} \subseteq G$.

- $\{* \, \bar{x}_i^2 : i = 1, \ldots, n \, *\} \subseteq ⊠_p \cup \{0\}$.
- Define $f: ⊠_p \to \mathbb{Z}$ via $f(a) = \#\{i: \bar{x}_i^2 = a\}$.
- **Fourier transform** of f: $\hat{f}(k) = \sum_{z \in ⊠_p} f(z)\, z^{-k} = \sum_{z \in \bar{R}} z^{-k} = S_{\frac{p-1}{2} - k}$
- **Inversion formula**: $f(z) \equiv \frac{2}{p-1} \sum_{k=0}^{(p-1)/2} \hat{f}(k)\, z^k \pmod{p}$, for $z \in ⊠_p$
- **Uncertainty Principle** (Feng, Hollmann & Xiang, 2019):

$$|\mathrm{supp}(f)| \cdot |\mathrm{supp}(\hat{f})| \geq |G|.$$

where $\mathrm{supp}(f) = \{x \in ⊠_p : f(x) \not\equiv 0 \bmod p\}$, $\mathrm{supp}(\hat{f}) = \{k: \hat{f}(k) \neq 0\}$. Hence,

$$|\{k: S_{2k} \neq 0\}| \geq \frac{p-1}{2n}.$$

Discrete Fourier Analysis

Example. $r = 3$, $n = 192$. $|G| = |B(192,3)| = 61 \cdot 155925 = 61 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 53$.

$$Q_{(n,r)}^{k}(\bar{x}_1, \cdots, \bar{x}_n) = c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \neq (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = \begin{cases} 0, & p - 1 \nmid 2k; \\ -m, & p - 1 | 2k. \end{cases}$$

Obtain: $S_2 = S_4 = \ldots = S_{2 \cdot 29} = 0$, $S_{2 \cdot 30} = 15$.

By **inversion formula**: $\#\{i \colon \bar{x}_i^2 = a\} = f(z) \equiv 31 \pmod{61}$

However, $31 \cdot \frac{p-1}{2} = 31 \cdot 30 > 192$. A contradiction!

Coefficients c_λ

Theorem (Xiao & Z. 2025+) For $k \geq 1$,

$$Q^k_{(n,r)}(X) = \sum_{\lambda\in\mathcal{P}(k)} \frac{(2k)!}{\prod_{i=1}^{\ell}(2\lambda_i)!}\cdot\frac{2^\ell}{\prod_{i=1}^{k} m_\lambda(i)!} \sum_{r_1+\cdots+r_{\ell+1}=r}\left(|B(n,r_{\ell+1})|\prod_{i=1}^{\ell} r_i^{2\lambda_i-1}\right)S_{2\lambda}$$

where $\mathcal{P}(k)$ stands for the set of all partitions of k, $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $m_\lambda(i) = \#\{i: \lambda_j = i, j = 1, \ldots, \ell\}$, $S_{2\lambda} = S_{2\lambda_1}\cdots S_{2\lambda_\ell}$ and $r_1, \ldots, r_{\ell+1} \in \mathbb{Z}_{\geq 0}$.

Proof (Sketch). Do NOT fix r. Try to prove

$$\sum_{r=0}^{\infty} Q^k_{(n,r)}(X)\, q^r = \frac{(1+q)^n}{(1-q)^{n+1}} \sum_{\lambda\in\mathcal{P}(k)} \frac{(2k)!}{\prod_{i=1}^{\ell}(2\lambda_i)!}\cdot\frac{2^\ell}{\prod_{i=1}^{k} m_\lambda(i)!}\prod_{i=1}^{\ell}\left(\sum_{s=1}^{\infty} s^{2\lambda_i-1}q^s\right)S_{2\lambda}$$

Set $S^n(r)$ to be the shell of $B(n,r)$. Then $Q^k_{S^n(r)}(X)$= $Q^k_{(n,r)}(X)$- $Q^k_{(n,r-1)}(X)$. Hence

$$\sum_{r=0}^{\infty} Q^k_{S^n(r)}(X)\, q^r = (1-q)\sum_{r=0}^{\infty} Q^k_{(n,r)}(X)\, q^r$$

Coefficients c_λ

Proof (continued). We only have to show

$$\sum_{r=0}^{\infty} Q^k_{S^n(r)}(X)\, q^r = \frac{(1+q)^n}{(1-q)^n} \sum_{\lambda\in\mathcal{P}(k)} \frac{(2k)!}{\prod_{i=1}^{\ell}(2\lambda_i)!} \cdot \frac{2^\ell}{\prod_{i=1}^{k} m_\lambda(i)!} \prod_{i=1}^{\ell}\left(\sum_{s=1}^{\infty} s^{2\lambda_i-1} q^s\right) S_{2\lambda}$$

Prove it by induction. Set

$$T(n+1,r) = \{\langle X, b\rangle : b \in S^{n+1}(r)\} = \{\pm b_1 X_1 \pm \cdots \pm b_{n+1} X_{n+1} : b_1 + b_2 + \cdots + b_{n+1} = r\}$$

$$\sum_{r=0}^{\infty} Q^k_{S^{n+1}(r)}(X)\, q^r = \sum_{r=0}^{\infty} \sum_{\alpha\in T(n+1,r)} \alpha^{2k}\, q^r$$

$$= \sum_{r=0}^{\infty} \sum_{\alpha\in T(n,r)} \alpha^{2k}\, q^r + \sum_{r=0}^{\infty} \sum_{\alpha\in T(n,r-1)} (\pm X_{n+1} + \alpha)^{2k}\, q^r + \cdots + \sum_{r=0}^{\infty} \sum_{\alpha\in T(n,0)} (\pm r X_{n+1} + \alpha)^{2k}\, q^r$$

To finish the proof: some tedious computation and properties of the Eulerian polynomial $A_n(q) = \sum_{w\in S_n} q^{des(w)}$:

$$A_{2k}(q) = (1+q) \sum_{\lambda\in\mathcal{P}(k)} \frac{(2k)!}{\prod_{i=1}^{\ell}(2\lambda_i)!} \cdot \frac{(2q)^{\ell-1}}{\prod_{i=1}^{k} m_\lambda(i)!} \prod_{i=1}^{\ell} A_{2\lambda_i-1}(q)$$

Coefficients c_λ

A **necessary condition** for the existence of lattice tiling of $B(n,r)$:

$$|G| = |B(n,r)| = pm, \text{ projection } \varphi: G \to (\mathbb{F}_p, +), \bar{x} := \varphi(x).$$

$$Q^k_{(n,r)}(\bar{x}_1, \cdots, \bar{x}_n) = c_{(2k)} S_{2k}(\bar{x}_1, \cdots, \bar{x}_n) + \sum_{\lambda \neq (2k)} c_\lambda S_\lambda(\bar{x}_1, \cdots, \bar{x}_n) = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

First Task: Determine the value of c_λ, and use them to get $\mathcal{S} := [S_2, S_4, S_6, \cdots]$ partially;

Example. For $n = 38$, $r = 3$, $|B(n,3)| = 43 \cdot 1771$.

$$\mathcal{S} = [0,0,0,0,0,\underset{6}{X_1},0,0,\ldots,\underset{21=\frac{p-1}{2}}{0},39,\ldots],$$

where X_1 is unknown.

Necessary Conditions

(#)--- $$c_{(2k)}S_{2k}(\bar{x}_1,\cdots,\bar{x}_n) + \sum_{\lambda\neq(2k)} c_\lambda S_\lambda(\bar{x}_1,\cdots,\bar{x}_n) = \begin{cases} 0, & p-1 \nmid 2k; \\ -m, & p-1 \mid 2k. \end{cases}$$

$$\mathcal{S} = [S_2, S_4, \ldots, S_{p-1}, \ldots],\ S_{2k} = \sum_{i=1}^{n} \bar{x}_i^{2k},$$

$$\mathcal{E} = [e_1, e_2, \ldots, e_n, \ldots],\ e_k = \sum \bar{x}_{i_1}^2 \bar{x}_{i_2}^2 \cdots \bar{x}_{i_k}^2.$$

List of Necessary Conditions:

1. $\exists\mathcal{S}$ fits (#), for instance Qureshi's criterion: $c_{(p-1)} \equiv 0 \not\equiv -m$, $c_{(2j)} \not\equiv 0$ for $2j < p-1$.
2. $\mathcal{S}$ must be of period $\frac{p-1}{2}$, and $S_{p-1} \leq n$
3. $e_{n+1} = e_{n+2} = \cdots = 0.$
4. $|\{i: \bar{x}_i = 0\}| < N_r \Rightarrow \max\{1 \leq k \leq n: e_k \neq 0\}$ should be large.

   (items 3–4) Only need to generate $S_2, \ldots S_{2n+\epsilon}$. Easy for $p \gg n$.
5. Uncertainty Principle (need large p): $|\{k: S_{2k} \neq 0\}| \geq \frac{p-1}{2n}$
6. Inversion Formula $f(z) \equiv \frac{2}{p-1}\sum_{k=0}^{(p-1)/2} S_{(p-1)/2-k}\, z^k$, check whether it determines a set of $\leq n$ nonzero squares in $\mathbb{F}_p$ and check $\{\sum \pm a_i\bar{x}_i : \sum a_i \leq r\} = m\mathbb{F}_p$

Necessary Conditions

Example 1. For $r = 3,\ \ 3 \le n \le 100$, Qureshi's criterion excludes

$$n = 6,12,21,39,48,64,66,75,93.$$

Example 2. For $r = 3, n = 26, |G| = 24857 = 7 \times 53 \times 67, p = 67, m = 371.$

$$S = [0, \ldots, 0, S_{2\times 15} = X, 0, \ldots, 0, S_{2\times 26} = 0, \ldots S_{2\times 33} = 33, \ldots]$$

$$S_{66} = \sum_{i=1}^{n} \bar{x}_i^{67-1} = 33 > 26$$

Example 3. For $n = 38, r = 3, |B(n,3)| = 43 \cdot 1771, p = 43$

$$\mathcal{S} = [0,0,0,0,0, X_1, 0,0, \ldots, 0,39, \ldots],$$

where X_1 is unknown.

$$e_{39} = 37X_1^3,\ e_{40} = e_{41} = 0,\ e_{42} = 39X_1^7 + 28.$$

$e_{39} = 0 \Rightarrow X_1^3 = 0$, but $e_{42} = 28 \neq 0.$

Necessary Conditions

Example 4. For $n = 11, r = 3, |B(n,3)| = 89 \cdot 23, p = 89$

$$S_2 = S_4 = \dots = S_{2\cdot n} = 0 = S_{2\cdot(n+1)} = \dots = S_{\frac{p-1}{2}-1}, S_{\frac{p-1}{2}-1} = 29$$

Hence $e_1 = e_2 = \cdots = e_n = 0$. A contradiction.

Example 5. For $n = 483, r = 3, |B(n,3)| = 155849 \cdot 967, p = 155849$.

The number of $S_{2i} = 0$ is more than $\frac{(n-1)\cdot(p-1)}{2n}$. A contradiction.

Necessary Conditions

Example 6.1. For $n = 107, r = 3, |B(n,3)| = 67 \cdot 43 \cdot 23 \cdot 5^2, p = 67$

$$S_{2\cdot 14} = X_0, S_{2\cdot 18} = X_1, S_{2\cdot 33} = 4, \text{ other } S_{2i} = 0 \text{ in one period}$$

where X_0, X_1 are unknown.

Compute $f(z) \equiv \frac{2}{p-1} \sum_{k=0}^{(p-1)/2} S_{(p-1)/2-k}\, z^k$. It NEVER defines a set of nonzero squares in $\mathbb{F}_p$ of size $\leq n$.

Example 6.2 (A nasty case). For $n = 84, r = 3, |B(n,3)| = 23^2 \cdot 13^2 \cdot 3^2, p = 23$

$$\mathcal{S} = [0,0,0,0,0,0,0, X_0, 0,0,0, \dots]$$

By Inversion Formula, the coefficients of elements in $\mathbb{F}_{23}$ are

$$[21X_0, 7X_0, 15X_0, 10X_0, 0, 5X_0, 0, 11X_0, 14X_0, 0, 0, 17X_0, 22X_0, 0, 0, 19X_0, 0, 20X_0, 0,0,0,0,0]$$

The ``Sum" of them is ≤ 23 if and only if $X_0 = 0$. Hence

$$\{* \, \bar{x}_i^2 : i = 1, \cdots, n \, *\} = \{* \, 23a^2, \, 23b^2, 23c^2, 15 \cdot 0 \, *\}$$

A complete search for a, b, c shows no solution for $\{\sum \pm a_i \bar{x}_i : \sum a_i \leq 3\} = m\mathbb{F}_p$.

4. Computational Results

Computational Results for $r = 3$ and $n \leq 1000$

- Extra Criterion [Zhang & Z. 2019] Assume that $n \equiv 1,5 \pmod 7$. If $24n + 1$ is not a square or $84 \nmid (24n + 1)^2 \pm 6\sqrt{24n + 1} + 29$, then no lattice tiling of $B(n, 3)$.
- We can exclude all $3 \leq n \leq 1000$ except for $35,\ 437,\ 590,\ 597,\ 805$.

n	Factorization of $\lvert B(n,3) \rvert$	Extra property
35	$29^2 \cdot 71$	$\exists p = 2n + 1$
437	$3 \cdot 5^4 \cdot 7 \cdot 47 \cdot 181$	
590	$3^2 \cdot 23 \cdot 1123 \cdot 1181$	$\exists p = 2n + 1$
597	$3^3 \cdot 5^2 \cdot 41 \cdot 43 \cdot 239$	
805	$3 \cdot 29^2 \cdot 179 \cdot 1543$	

Unsolved cases

- G, $|G| = |B(n,r)| = pm$, $\varphi: G \to (\mathbb{F}_p, +)$, $R = \{x_1, \ldots, x_n\}$
- $\varphi(R) = a\{0\} \cup b\mathbb{F}_p^*$, $a + (p-1)b = n$ and there is no contradiction.
- If it happens for every prime $p||G|$, then it is impossible to prove the nonexistence of lattice tiling of $B(n,r)$ using this approach.
- We need to consider projection from G to $\mathbb{Z}_{p_1^{i_1}} \times \cdots \times \mathbb{Z}_{p_s^{i_s}}$ (instead of $\mathbb{F}_p$). (No Fourier Transforms) How to do it?

Concluding Remarks

- This approach also works for
  - other r,
  - lattice packings with density ≈ 1 (for instance, the almost perfect case).
- One main difficulty for large r and n:

$$\text{factorize } |B(n,r)| = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{n}{i}\binom{r}{i}$$

- We can also prove the nonexistence of lattice tiling of $B(n,r)$ with fixed r for infinitely many n, for instance:
  - $r = 6,\ n \equiv 10, 22, 35, 47, 60, 72, 97, 110, 122 \pmod{5^3}$, red ones are new compared with using Qureshi's criterion, because we can completely determine the coefficients of $Q^k_{(n,r)}(X)$.

Concluding Remarks

Some connections not mentioned:

- (Degree-Diameter Problem in **graph theory**) Abelian Cayley graph meeting the **Moore-like bound**⇔Lattice tiling of $B(n, r)$
- (Lattice) Tiling of $\mathbb{Z}_q^n$ by Lee spheres (Weak GW Conjecture)
- Association Schemes for Lee metric in $\mathbb{Z}_q^n$: multivariate **Lloyd polynomials**

"For $r = 2, T^2 = 2G - T^{(2)} + 2ne \in \mathbb{Z}[G]$, project T to a subset in $(\mathbb{F}_p, +)$ and use the multiplicative character $z \mapsto z^k$."

- **One more question**: Can we use the same trick for (generalized) difference sets, (near) factorization of groups, etc?

Thanks for your attention!