New Combinatorial Structures in Projective Planes of Order 16

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Outline

Definitions

A 2-(v, k, λ) design $D = \{X, \mathcal{B}\}$ is a pair of a **point set** X of size v and a collection of k-subsets $\mathcal{B}=\{B_j\}_{j=1}^{b}$ called **blocks**, such that every pair of points is contained in exactly λ blocks.

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A design is *resolvable* if it admits a resolution.

Maximal Arcs

Let Π be a finite projective plane of order $q = gk$.

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If $\mathcal{A}$ is a maximal $\{v; k\}$-arc, then every line of Π is either disjoint from $\mathcal{A}$, or meets $\mathcal{A}$ in k points.

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The set of lines of Π which have no points in common with $\mathcal{A}$ determines a maximal $\{(k(g-1)+1)g; g\}$-arc $\mathcal{A}^{\perp}$ in the dual plane $\Pi^{\perp}$.

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Let D be a resolvable $2-(mk, k, 1)$ design.

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Two resolutions, R_1, R_2 of D,

$$R_1 = P_1^{(1)} \cup P_2^{(1)} \cup \cdots P_r^{(1)},\ R_2 = P_1^{(2)} \cup P_2^{(2)} \cup \cdots P_r^{(2)},$$

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A set of resolutions $\mathcal{R} = \{R_1, R_2, \cdots R_n\}$ is called compatible if every pair of resolutions in $\mathcal{R}$ is compatible.

An Upper Bound

Theorem 1 (Tonchev 2017)

Let $\mathcal{R} = \{R_1, R_2, ..., R_n\}$ be a set of n mutually compatible resolutions of a $2-(v,k,1)$ design D, where $v = (g(k-1)+1)k$.

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$$n \leq (k(g-1)+1)g.$$

Equality holds if and only if there exist a projective plane Π of order $q = gk$ such that D is embeddable in Π as a maximal $\{v;k\}$-arc.

In [6], Penttila et al did a computer search and classified all degree 2 and 8 maximal arcs in the known planes of order 16.

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In [1], Ball and Blokhuis proved that up to isomorphism PG(2,16) contains only two maximal (52,4)-arcs.

#	Maximal (52,4)-arc	Design's Aut Group Order	2-rank of designs	# of parallel classes	# of resolutions	# of compatible resolutions	Plane(s) isomorphic to?
1	*desg.1*	68	41	2329	409	52 (×1)	**desg**
2	*desg.2*	408	41	2550	460	52 (×1)	**desg**
3	*demp.1*	*24*	*49*	*250*	*52*	*52* (×1)	**demp**
4	*demp.2*	*144*	*47*	*543*	*52*	*52* (×1)	**demp**
5	*semi4.1*	*96*	*45*	*2569*	*52*	*52* (×1)	**semi4**
6	*semi2.1*	*24*	*47*	*327*	*52*	*52* (×1)	**semi2**
7	*semi2.2*	*144*	*45*	*1279*	*55*	*52* (×1)	**semi2**
8	*lmrh.1*	*96*	*47*	*2265*	*104*	*52* (×2)	**lmrh** and **lmrh**$^\perp$
9	*math.1*	*24*	*49*	*291*	*52*	*52* (×1)	**math**
10	*hall.1*	*24*	*49*	*274*	*52*	*52* (×1)	**hall**
11	*bbh1.1*	*24*	*47*	*330*	*52*	*52* (×1)	**bbh1**
12	*bbh1.2*	*32*	*46*	*2017*	*136*	*52* (×2)	**bbh1** and **john**
13	*jowk.1*	*16*	*46*	*1389*	*52*	*52* (×1)	**jowk**
14	*jowk.2*	*32*	*46*	*2409*	*104*	*52* (×2)	**jowk** and **john**
15	*john.1*	*32*	*47*	*1953*	*144*	*52* (×2)	**john**
16	*john.2*	*32*	*47*	*1953*	*144*	*52* (×2)	**john**
17	*dsfp.1*	*24*	*47*	*1045*	*52*	*52* (×1)	**dsfp**

Table 1: The maximal (52,4)-arcs and their designs

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5	*demp.3*	*24*	*47*	*611*	*52*	*52* (×1)	**demp**
6	*demp.4*	*48*	*47*	*531*	*52*	*52* (×1)	**demp**
7	*semi4.1*	*96*	*45*	*2569*	*52*	*52* (×1)	**semi4**
8	*semi2.1*	*24*	*47*	*327*	*52*	*52* (×1)	**semi2**
9	*semi2.2*	*144*	*45*	*1279*	*55*	*52* (×1)	**semi2**
10	*semi2.3*	*32*	*45*	*1497*	*52*	*52* (×1)	**semi2**
11	*semi2.4*	*32*	*45*	*1313*	*52*	*52* (×1)	**semi2**
12	*semi2.5*	*16*	*46*	*1045*	*52*	*52* (×1)	**semi2**
13	*semi2.6*	*48*	*47*	*547*	*52*	*52* (×1)	**semi2**
14	*semi2.7*	*48*	*45*	*691*	*52*	*52* (×1)	**semi2**
15	*lmrh.1*	*96*	*47*	*2265*	*104*	*52* (×2)	**lmrh** and **lmrh**$^{\perp}$
16	*lmrh.2*	*32*	*47*	*2377*	*64*	*52* (×1)	**lmrh**
17	*math.1*	*24*	*49*	*291*	*52*	*52* (×1)	**math**
18	*math.2*	*32*	*46*	*1729*	*52*	*52* (×1)	**math**
19	*math.3*	*32*	*47*	*2401*	*64*	*52* (×1)	**math**
20	*math.4*	*32*	*46*	*1665*	*52*	*52* (×1)	**math**
21	*math.5*	*16*	*47*	*1233*	*52*	*52* (×1)	**math**
22	*math.6*	*16*	*48*	*1329*	*52*	*52* (×1)	**math**
23	*math.7*	*16*	*48*	*1125*	*52*	*52* (×1)	**math**
24	*hall.1*	*24*	*49*	*274*	*52*	*52* (×1)	**hall**
25	*bbh1.1*	*24*	*47*	*330*	*52*	*52* (×1)	**bbh1**
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31	*john.3*	*32*	*46*	*2017*	*136*	*52* (×2)	**john** and **bbh1**
32	*john.4*	*32*	*46*	*2409*	*104*	*52* (×2)	**john** and **jowk**
33	*dsfp.1*	*24*	*47*	*1045*	*52*	*52* (×1)	**dsfp**

Table 2: The maximal (52,4)-arcs and their designs

Theorem 2

The number of pairwise non-isomorphic resolvable $2-(52,4,1)$ designs is greater than or equal to 50, previous to our work this bound was 30.

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Theorem 3

The number of maximal arcs of degree 4 in planes of order 16 is greater than or equal to 63, previous to our work this bound was 32.

Codes of Maximal Arcs

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The parameters and the order of the automorphism group of $C(D)$ are computed for all known maximal arcs in the planes of order 16.

Furthermore, using Magma [3], codes were sorted according to their weight distributions, and codes having the same weight distributions were tested for equivalences.

No.	2-rank	(52,4)-Arc	(A_2,A_4)	$\lvert Aut(C(D))\rvert$
1	41	desg.1	(0,221)	$2^{2}17^{1}$
2	41	desg.2	(0,221)	$2^{3}3^{1}17^{1}$
3	43	hall.1$^{\perp}$	(6,1037)	$2^{6}3^{2}$
4	45	demp.1$^{\perp}$	(24,3989)	$2^{33}3^{2}$
5	45	{demp.2$^{\perp}$, semi2.2}	(6,4325)	$2^{18}3^{4}5^{1}7^{1}$
6	45	{semi4.1, semi2.7}	(0,4469)	$2^{17}3^{3}$
7	45	semi2.3	(18,4165)	$2^{26}3^{1}$
8	45	semi2.4	(16,4277)	$2^{25}3^{1}$
9	46	john.3	(42,8293)	$2^{38}3^{5}$
10	46	{john.4, jowk.1, math.2}	(26,8613)	$2^{37}3^{2}$
11	46	jowk.2$^{\perp}$	(46,8325)	$2^{38}3^{6}5^{1}$
12	46	{math.4, math.4$^{\perp}$}	(42,8549)	$2^{38}3^{5}$
13	46	math.5$^{\perp}$	(42,8549)	$2^{36}3^{4}$
14	46	semi2.5	(50,8453)	$2^{37}3^{6}$
15	47	bbh1.1	(120,16853)	$2^{40}3^{13}5^{3}7^{3}$
16	47	{demp.2, demp.4}	(72,17045)	$2^{44}3^{14}$
17	47	{dsfp.1, demp.3}	(74,16997)	$2^{45}3^{13}$
18	47	dsfp.1$^{\perp}$	(66,17093)	$2^{43}3^{11}$
19	47	{john.1, lmrh.1, lmrh.2, lmrh.2$^{\perp}$, math.2$^{\perp}$, math.3, math.3$^{\perp}$, math.6$^{\perp}$, math.7$^{\perp}$, semi2.1, semi2.6, demp.4$^{\perp}$}	(78,16901)	$2^{45}3^{15}$
20	47	jowk.1$^{\perp}$	(94,16709)	$2^{44}3^{14}5^{1}7^{1}$
21	47	math.5	(106,16869)	$2^{43}3^{12}5^{2}7^{2}$
22	47	demp.3$^{\perp}$	(98,16965)	$2^{41}3^{11}5^{2}7^{2}$
23	48	{john.1$^{\perp}$, john.2$^{\perp}$, john.3$^{\perp}$, john.4$^{\perp}$, math.6, math.7}	(174,33669)	$2^{48}3^{14}5^{6}7^{6}$
24	49	{hall.1, math.1, math.1$^{\perp}$}	(366,67205)	$2^{49}3^{20}5^{9}7^{6}11^{3}13^{3}$
25	49	demp.1	(408,67541)	$2^{46}3^{19}5^{9}7^{6}11^{3}13^{3}17^{3}$

Table 3: $2-(52,4,1)$ designs and their codes

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4. Codes associated with desg.1, desg.2 and semi4.1 have minimum distance 4, while the minimum distance of all other codes is 2.

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4. Codes associated with desg.1, desg.2 and semi4.1 have minimum distance 4, while the minimum distance of all other codes is 2.
5. The code of semi4.1 is optimal: it has the largest possible minimum distance for the given length 52 and dimension 45.

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4. Codes associated with desg.1, desg.2 and semi4.1 have minimum distance 4, while the minimum distance of all other codes is 2.
5. The code of semi4.1 is optimal: it has the largest possible minimum distance for the given length 52 and dimension 45.
6. The 50 codes are partioned into 25 equivalence classes.

	DESG	DEMP	SEMI4	SEMI2	LMRH	MATH	HALL	BBH1	JOWK	JOHN	DSFP	BBH2	BBS4
DESG							1						
DEMP				1,5	5	5	5		3	5	2,5		
SEMI4			2	3,5	1				1	1	1		1
SEMI2		1,5	3,5		5	5				5			
LMRH		5	1	5	4,5	5			2	5	3		
MATH		5		5	5	5	5		2,5	5			
HALL	1	5				5		1		1		1	
BBH1							1			4,5			
JOWK		3	1		2	2,5				4,5			
JOHN		5	1	5	5	5	1	4,5	4,5				
DSFP		2,5	1		3								
BBH2							1						
BBS4			1										

Table 4: Connections between planes of order 16

Thank you!

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